ON l_∞ PERFORMANCE OPTIMIZATION: LINEAR SWITCHED SYSTEMS AND SYSTEMS WITH CONE CONSTRAINTS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mechanical Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2016

Urbana, Illinois

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Abstract

The l_∞ performance of Linear Time-Invariant (LTI) systems has been one of the corner stones of the robust control theory for over the past 30 years. The l_∞ performance has been studied mostly for LTI systems and the scarcity of the results for other types of systems is prominent in this area. This dissertation aims to depart from LTI systems and investigate the l_∞ performance for other classes of systems. In particular, the l_∞ performance of Linear Switched Systems (LSS) and of linear systems with cone constraints is studied in the first and second part of this dissertation, respectively.

Part I: In Part I, we first consider the worst-case l_∞ induced norm computation of LSS. That is, $\sup_\sigma \|G_\sigma\|$, where G_σ is a LSS, σ is the switching sequence, and the norm, $\|.\|$, is the l_∞ induced norm. This problem can be linked to robustness of systems when the switching is arbitrary. We provide lower and upper bounds of this quantity. These bounds are hard to compute and in general conservative. Hence, we narrow our attention to special classes of LSS by defining the classes of input, output, and input-output LSS and show that for these classes, exact expressions for the worst-case l_∞ induced norm can be found. Moreover, we introduce the class of generalized input-output LSS and show how their l_∞ gains can be computed exactly via Linear Programming (LP). The class of generalized input-output LSS proves to be a sufficiently rich class as it is dense in the set of all stable LSS. We further derive new stability and stabilizability conditions and control synthesis in terms of LP utilizing generalized input-output LSS.

The other extreme from the worst-case norm is the minimal norm, i.e., $\inf_\sigma \|G_\sigma\|$. The interest in this type of problem is motivated by situations where there may be limited sensor and/or actuator resources for filtering and control. We show that for Finite Impulse Response (FIR) switching systems the minimizing switching sequence can be chosen to be periodic. For input-only or output-only switching systems an exact characterization of the minimal l_∞ gain is provided, and it is shown that the minimizing switching sequence is constant, which, as also shown, is not true for input-output switching.

Moreover, we study Markov Linear Switched Systems (MLSS). These are LSS whose switching sequence is a Markov process. We introduce the notion of the stochastic l_∞ gain and provide exact expression to compute it. However, this computation is challenging, as we show, and hence we resort to a more relaxed but tractable notion of l_∞ mean performance. We provide tractable computation and control synthesis method with respect to the l_∞ mean performance.

Part II: Part II of this dissertation deals with the l_∞ gain of linear systems with positivity type of constraints. The study of such systems is well justified as there are many physical problems in which some variables are restricted be non-negative (or non-positive); examples can be found in biology, economics, and many other areas. We consider the case when the output is forced to be in the positive l_∞ cone when the input is in this cone. This reflects as, so-called, an external positivity constraint on the system. As we point out, if such a constraint is imposed on the closed loop map, finding an optimal controller is LP and hence a tractable problem. If, on the other hand, the constraint known as internal positivity is sought, we show that a dynamic controller offers no advantage over a static one. These results can be used to obtain an optimal (static) state feedback controller. However, designing an optimal output feedback controller (which is static) is a harder problem and in general leads to a bilinear program. We show that this bilinear program can be reduced to LP, if the null space of the measurement matrix is invariant under multiplication by diagonal matrices.

Besides the positive systems mentioned above, we consider the case where only the input is restricted to be in the positive cone of l_∞, denoted by l_∞^+, and seek to characterize the induced norm from l_∞^+ to l_∞. We stress here that no positivity constraint is imposed on the system itself. As an example, consider a positive nonlinear system with positive input that is linearized about a point other than origin. The linearized model is no longer a positive system as it is not linearized about the origin. Its inputs, however, remain positive and hence fit into this class of problems. We obtain an exact characterization of this norm (the induced norm from l_∞^+ to l_∞) which can be used to synthesis a controller minimizing the induced norm from l_∞^+ to l_∞ via LP.

To my parents, MohammadReza and Mehri...

Acknowledgments

This is the last piece of this dissertation that I am writing. This is the end of the journey that took place over the past few years, 2010-2015. There is a number of people whom I would like to thank as they have been most influential during this period and this dissertation would not have been possible without their help.

First and foremost, I would like to express my deepest gratitude to my adviser, Professor Petros Voulgaris. I consider myself privileged working under his supervision. I have the utmost respect for his mentorship, support, intelligence, and patience. I also admire his sense of humor and specially his sarcastic comments. Our countless number of meetings were truly enlightening and helpful. On the contrary to what some may believe or experience, my PhD life at the university of Illinois was not stressful at all, yet intellectually challenging. To a great extent, I owe this to the support and guidance of my adviser. His patience with me provided a great environment for me to conduct my research with a peace of mind. I would also like to thank Professor Geir Dullerud whom supported me for a period of time. Our collaboration, although short, was very fruitful.

I would like to thank my committee members, Professors Naira Hovakimyan and Tamer Basar. Thank you for taking time to review this dissertation and providing me with useful comments. Also, special thanks to Naira for accommodating me in her lab for one year when I first started at the University of Illinois.

I believe that anyone with a graduate school experience would agree with me on the very fact that the academic life of a graduate student is intertwined with his/her personal life. With that, I want to thank my family and the circle of friends. Special thanks to my wife, Negin, for her love and support. Also, during my stay in Champaign-Urbana, I was fortunate for making great friends from all around the world who made my time here quite memorable. I learned a lot from them and want to thank them all! I am especially thankful to my "coffee-shop working" companions including Negin. A great deal of my research and this dissertation is the fruit of the days and nights we all worked together and motivated each other. Without them and the joy of our game breaks, my graduate life would not be as fulfilling as it turned out.

I would like to extend my gratitude to my Volleyball friends. Academic life gets challenging at times. You never know if your research will land where you like, if the journal reviewers will eventually accept your revisions, if you can present at your favorite conference, or if, after all of these, you can do something with your Ph.D. that makes you happy for the rest of your life. Like all of my graduate student fellows, I was facing these ambiguities and concerns everyday. Our volleyball games always helped me to de-stress, regather my energy, and get back on track.

This would not be possible without my great team-mates who all generously brought their energy, positivity, and enthusiasm to our games every week.

And last but not the least, I would like to thank all the professors with whom I took great courses and from whom I learned a lot. I also like to thank the staff of the mechanical engineering department and Coordinated Science Laboratory for kindly helping me with all the nuances the administrative procedures during my Ph.D.

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Chapter 1

Introduction

1.1 On l_∞ Performance: a Historical Overview

The l_∞ performance of Linear Time-Invariant (LTI) systems has been an important complement to l_2 performance and a significant aspect in the development of robust control theory over the past 30 years. While several developments in control with quadratic type of criteria have been extended to other types of systems such as Linear Switched Systems (LSS) and positive systems, it is not the case for l_∞ criteria. This is what this dissertation aims to achieve. In particular, the first part of this work is devoted to the l_∞ performance of LSS, and in the second part, we provide new results on systems with positive cone constraints.

Figure 1.1 depicts the general setup of a control system. Therein, G is the nominal generalized plant, K is the controller, Δ is the uncertainty block, w is the exogenous input, and z is the regulated output. Most of the control problems can be reduced to this form where the objective is to design a controller K such that it robustly stabilizes the plant for all admissible Δ's while minimizing the effects, measured in a certain metric, of the exogenous input on the regulated output. The l_1 control theory developed to deal with persistent but bounded disturbances. The rejection of l_∞ disturbances was first formulated in [1]. Back then, approaches were only available to deal with two types of exogenous inputs. Either the exogenous input was somehow known, e.g. sinusoid or step, or it was assumed to be square-integrable, i.e. l_2 signal. In either case, the objective was to minimize the maximum (weighted) energy of the output. [1] was a genuine paper as it gave birth to the theory of l_1 robust control but did not provide a

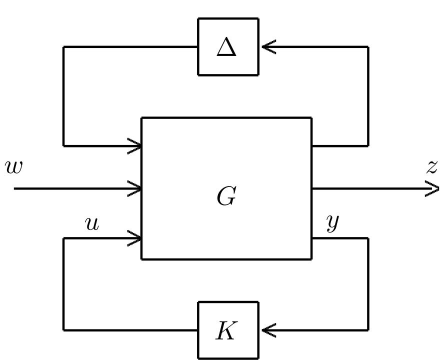


Figure 1.1: General setup of a control system

complete solution. The l_1 control problem was first fully solved in [2] for SISO systems. It was shown that the optimal controller can be retrieved from a solution to a finite dimensional Linear Program (LP). Furthermore, the MIMO case was considered and solved in [3]. Later, in [4] and [5], the l_1 control theory proved to be a cornerstone of the robust control together with the $\mathcal{H}_\infty$ theory. The l_1 theory gained popularity not only because it allows one to cope with persistent disturbances but also finding the optimal solution is computationally efficient as it reduces to LP. Another appealing feature of the l_1 (or l_∞) framework is that the time-domain constraints, such as saturation, can be handled directly whereas this is not the case in the l_2 setting.

The common approach in finding an l_1 optimal controller is to formulate the problem as a Model-Matching Problem (MMP). This can be done through the Youla-Kucera parameterization [6]; that is parameterizing the set of all stabilizing controllers by a stable parameter Q which is referred to as the Youla parameter. Then, the set of all closed-loop maps can be written as affine maps in the Youla parameter and consequently the problem reduces to

$$\inf_{Q} \|H - UQV\|, \tag{1.1}$$

where H, U, and V are stable systems defined in terms of the plant parameters and $\|.\|$ is the l_∞-induced (l_1) norm. A few years after this problem solved in [3], it was also shown that a time-varying or a smooth nonlinear control offers no advantage over LTI one [7], [8]. Also, it was shown that even in the case of full state feedback, unlike the $\mathcal{H}_\infty$ optimal control, the l_1 optimal control can be dynamic of arbitrarily high order [9]. However, invoking viability theory, it is proved [10] that there exists an optimal static (possibly non-smooth) nonlinear controller for the full state feedback. Moreover, the author presents a constructive algorithm for such a controller in [11]. On the subject of l_∞ filters, one can refer to [12] where the author addresses the problem of minimizing the worst-case magnitude of the estimation error over unknown but l_∞ input signals. The problem is essentially formulated as a MMP and solved via LP.

Aside for the LTI systems, the l_1 control theory for other classes of systems is not studied very much. Of the few extensions departing from LTI, one can refer to [13] and [14]. In the former, it was shown that the performance of slowly time-varying systems cannot be drastically different from that of the time-invariant frozen-time systems. In [14], the multirate and periodic systems are studied. The authors use lifting techniques to reduce the problem to time-invariant with additional constraints imposed on the controller to ensure causality. It was shown that control synthesis for periodic systems is not much different than the standard l_1 control problem.

1.2 Linear Switched Systems

Linear Switched Systems (LSS) are a special class of hybrid systems and have been the subject of many studies over the last twenty years or so. Researchers have focused on many aspects of such systems. We refer to [15], [16], [17], and references therein for some of the works done in this area. LSS can be used to model various practically important situations and hence deserve a thorough study. They can be used to model systems with sudden parameter variations, sudden change of system structure due to various reasons such as failures, lossy communications, etc.

In the literature, stability analysis and stabilizability of such systems have been given a major attention. One can refer to [18], [19], [20], [21], and [22] regarding the stability analysis and stabilizability conditions for switched linear systems. Unfortunately, these conditions are all combinatorial and computationally hard to check. Indeed, the stability of a switched linear system is, in general, an undecidable problem [23]. As a trade-off, one can consider the tractable but sufficient conditions such as quadratic stabilizability, or the existence of a common Lyapunov function.

Similar to LTI systems, input-output properties of LSS are important. A relevant question is what the different gains of a LSS are and how they are possibly related to the gains of the LTI modes of this system. There are works such as [24], [25], [26], [27], [28], and [29] that deal with finding the quadratic type of performance for such systems. In [26], the worst-case $\mathcal{L}_2$ induced gain of a LSS is studied when the switching is slow and the time between two consecutive switches approaches infinity. In the case of slow switching (when the dwell time approaches infinity), on the contrary to what one might expect, the gain of the switched system can be, in general, arbitrarily larger than that of its LTI modes. It is argued that the worst-case switching scenario suffices to have one switch when the dwell time approaches infinity. In [29], the $\mathcal{L}_2$ induced norm of periodic LSS is studied in the case of fast switching (when the rate of switching approaches infinity). It was shown that the $\mathcal{L}_2$ induced norm of a fast switching LSS is in general different than that of the average system. The authors defined the term input-output energy gain of the system and showed for a fixed $\mathcal{L}_2$ input signal, if only the state coefficient matrix switches, the input-output energy gain of a LSS approaches the $\mathcal{L}_2$ gain of the average system as the rate of switching grows to infinity. In [28], an H_2 type of cost is studied and upper and lower bounds are provided for continuous as well as discrete-time systems.

In stochastic frameworks, Markov Linear Switched Systems (MLSS) have been studied in a large body of literature, e.g., [30], [31], [32], and [33]. A MLSS is a LSS whose switching law is a Markov process. As an example, the packet delivery of a network can be modeled as a Markov process and combined with the LTI plant results in a MLSS. Most of the literature on the input-output properties of LSS and MLSS are analyzed in quadratic setting. In the context of l_1 or l_∞ induced gains, very little has been done. This is what we address in the first part of this dissertation.

1.3 Systems with Cone Constraints

There are many dynamical systems in which some variables are restricted to be non-negative (or non-positive); examples can be found in biology, economics, and many other areas [34], [35], [36]. Motivated by such problems, the theory of positive systems has been the focus of many researchers. Notions such as stability, stabilizability, positive realization, and (distributed) control synthesis of such systems have been the subject of research, see e.g. [37], [38], [39], [40].

For linear systems, the notion of internal positivity refers to the case when the states of the system remain nonnegative if the inputs and the initial conditions are nonnegative. Many aspects of positive linear systems have been investigated extensively, see for example [41]. The controllability of linear positive systems is studied in [42]. The problem of positive realization is considered in [43] and [44]. The input-output properties, and in particular the gains of such systems, have also been given major attention in [45], [46], [47], and references therein. In [45], copositive linear Lyapunov functions and linear supply rates are used, in the context of dissipativity theory, to investigate robust stability and performance. Further, the problem of synthesizing an optimal l_∞-induced static state-feedback controller with given sparsity or boundedness constraints is considered and solved. Synthesizing an optimal l_1-induced static state-feedback controller is studied in [46] and [48]. In the latter, the problem is reduced to a bilinear program and an iterative algorithm is utilized to solve it. The output feedback, however, is a more challenging problem. This problem, in general, can be cast as a bilinear program and in certain cases, it can be reduced to a linear program. In [49], a linear program is provided to find a rank one static output-feedback gain such that the closed loop system is stable and internally positive. For l_2 type of performance, one can refer to [50], [51], and [47].

Aside from positive systems whose states and outputs are positive, there are types of systems with cone constraints. For example, one can think of a not necessarily positive system whose input is restricted to be positive. As an example, consider a positive nonlinear system with positive input that is linearized about a point other than origin. The linearized model is no longer a positive system as it is not linearized about the origin. Its inputs, however, remain positive and hence fit into this class of problems. To the best of our knowledge, the input-output properties of this type of systems are not considered before although they deserve theoretical investigation. In the second part of this dissertation, we develop novel results on positive systems as well as the on the less studied systems with positive input.

1.4 Contribution of the Dissertation

As mentioned above, the l_1 theory is mainly limited to LTI systems and the extensions address slowly time-varying and periodic systems. We extend this theory to the classes of LSS, MLSS, and general LTV systems, in the first part of this dissertation.

In Chapter 2, Section 2.2, we consider the worst-case l_∞-induced norm computation of LSS. This is a highly complex problem. In fact, a prerequisite to compute the gain of LSS is stability, which is an undecidable problem [23]. Therefore, for the sake of well-definedness, we assume that the LSS is stable when computing the norm. We find bounds on the worst-case l_∞-induced norm and discuss how finding those bounds is a complicated task. This, indeed, is not surprising due to the complexity of the class of LSS. Therefore, we restrict our study to the subsets of LSS whose gain computation can be done more efficiently. To this end, we introduce the classes of *output switching systems*, *input switching systems*, *input-output switching systems*, and *generalized input-output switching* *systems.* For these classes of LSS, we provide exact expressions to compute the worst-case gain via LP. The interest in these classes of LSS is not only because their gain can be computed efficiently but also they can be used to model interesting practical situations, for example switching between actuator/sensors. Furthermore, as we show, the class of generalized input-output switching systems is dense in the space of all stable LSS. We use this fact to derive new stability condition for LSS in terms of MMP which can be cast as LP. Furthermore, utilizing generalized input-output switching systems, we compute the gain of a general LSS and can synthesize optimal controllers.

Next, in Section 2.3, we consider the problem of computing the minimal gain. That is, we try to answer the question of what switching law results in the smallest l_∞-induced norm. We show that an optimal switching is periodic. This relates to the sensor scheduling problems. Furthermore, for a periodic switching, one can employ lifting techniques and design a filter/controller for the invariant representation of the system similarly to [14].

In Chapter 3, we study the Markov Linear Switched (MLSS) systems. These are LSS where the switching sequence is a Markov process. To study these systems, we define the notion of the stochastic l_∞ gain and show how it can be computed. Moreover, we study the mean performance of MLSS and present an optimal controller synthesis to minimize the l_∞ gain of the mean representation.

In the second part of this dissertation, we study the l_∞ performance and control design of the LTI systems subject to positivity constraints. More precisely, we study two types of systems, the LTI systems whose inputs are restricted to the positive cone of l_∞ and positive LTI systems.

In Chapter 4, the (not necessarily positive) systems with positive inputs are studied. We introduce the notion of *plus norm* to characterize the input-output gain of such systems. The plus norm is defined to be the induced norm of the system from the positive cone of l_∞ to l_∞. We provide exact computation of the plus norm in terms of the l_∞ and the DC gain of the system which can be performed via LP. This can be used to synthesize optimal plus norm controllers. Using duality theory, we further show that the optimal plus norm controller exhibits certain features similar to those of the standard l_1 optimal controller. More precisely, for one-block problems, both controllers can be found through finite dimensional LP in the dual space and both result in FIR closed-loop. Moreover, we show that a smooth time-varying nonlinear controller cannot outperform a LTI controller, in the plus norm sense.

In Chapter 5, two notions of positive systems are considered, external and internal. Externally positive systems are the systems whose outputs remain nonnegative as long as the inputs are nonnegative. A system is said to be internally positive if, in addition to the outputs, the states remain nonnegative when the initial condition and inputs are nonnegative. Examples of such systems arise naturally in economics, biology, etc. We show how synthesizing a controller enforcing closed-loop external positivity can be cast as LP. Furthermore, if the internal positivity of the closed-loop is desirable, we argue that a dynamic controller offers no advantage over a static one. Finding the static controller turns out to be a bilinear program, in general. However, in certain cases, the problem reduces to LP. As shown, these are the cases of full or partial state feedback, or if the measurement matrix (C-matrix) is invariant under multiplication by diagonal matrices.

To summarize, the contributions of this work are the following:

- Worst-case l_∞ gain computation for certain classes of LSS; these are input, output, input-output, and generalized input-output switching systems.
- Worst-case l_∞ gain computation for general LSS through approximation with generalized input-output switching systems.
- Minimal l_∞ gain computations.
- The extension of the l_1 optimal control theory to LTV and LSS in both deterministic and stochastic frameworks.
- l_∞ analysis and control synthesis for externally and internally positive systems:
- l_∞ analysis and control synthesis for systems with positive inputs.

1.5 Some Generic Notation

In this section we define the notation used throughout this dissertation. By $\mathbb{R}$ and $\mathbb{Z}$ we mean the sets of real numbers and integers, respectively. We further use $\mathbb{Z}_+$ to denote the set of non-negative integers. The set of n-tuples $x = \{x(k)\}_{k=0}^{n-1}$ where $x(k)$s are real numbers is denoted by $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, its l_∞ and l_1 norm are defined as

$$\begin{aligned} \|x\|_\infty &= \max_{k \in \{0,1,\ldots,n-1\}} |x(k)|, \\ \|x\|_1 &= \sum_{k=0}^{n-1} |x(k)|. \end{aligned}$$

Let $g = \{g(k)\}_{k=0}^{\infty}$ be a sequence where $g(k) \in \mathbb{R}^n$. Then, the l_∞ and l_1 norm of this sequence are defined as

$$\begin{aligned} \|g\|_\infty &= \sup_{k \in \mathbb{Z}_+} \|g(k)\|_\infty, \\ \|g\|_1 &= \sum_{k=0}^{\infty} \|g(k)\|_1, \end{aligned}$$

whenever they are finite. The set of sequences whose l_∞ norm (l_1 norm) is finite is denoted by l_∞^n (l_1^n). We may use l_∞ (l_1) instead of l_∞^n (l_1^n) when the dimension n is clear from context or not important. For a sequence $g = \{g(k)\}_{k=0}^{\infty} \in l_\infty$, its λ-transform is defined as

$$\hat{G}(\lambda) = \sum_{k=0}^{\infty} g(k) \lambda^k,$$

for the values of λ that the summation converges. Given a linear operator (or matrix) $T : l_\infty \to l_\infty$ ($T : \mathbb{R}^n \to \mathbb{R}^m$), its l_∞ induced norm, $\|T\|$, is defined as

$$\|T\| := \sup_{f \neq 0} \frac{\|Tf\|_\infty}{\|f\|_\infty}.$$

It can be easily verified that for a finite dimensional matrix $X : \mathbb{R}^n \to \mathbb{R}^m$,

$$\|X\| = \sup_i \sum_{j=1}^{n} |x_{ij}| ,$$

where x_{ij} is the entry at row i and column j of X. The standard truncation and delay operators are denoted by Π and Λ, respectively. More precisely, for any $k \in \mathbb{Z}_+$ and any sequence $g = \{g(0), g(1), ...\}$,

$$\begin{aligned} \Lambda^k g &= \left\{ \underbrace{0, ..., 0}_{k \text{ zeros}}, g(0), g(1), ... \right\}, \\ \Pi^k g &= \{g(0), g(1), ..., g(k-1), 0, 0, ...\}, \end{aligned}$$

and $\Lambda^{-k} g = \{g(k), g(k+1), ...\}$.

Part I

Linear Switched Systems

Chapter 2

Deterministic Linear Switched Systems

2.1 Background

In general, a LSS is composed of finite number of LTI subsystems and a rule that orchestrates switching between subsystems. In state-space, a LSS H_σ can be realized as

$$H_\sigma : \begin{cases} x(t+1) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) \\ y(t) = C_{\sigma(t)} x(t) + D_{\sigma(t)} u(t) \end{cases} , \tag{2.1}$$

where $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_N := \{1, 2, ..., N\}$ is referred to as the switching sequence and the 4-tuples $\left(A_{\sigma(k)}, B_{\sigma(k)}, C_{\sigma(k)}, D_{\sigma(k)}\right)$ assumes values in the set

$$\{(A_i, B_i, C_i, D_i) : i \in \mathbb{Z}_N\},$$

for $k \in \mathbb{Z}_+$. Sometimes, σ is restricted to be in the set of admissible switching sequences. We denote this set by Ξ which is a subset of all N valued sequences. Each LTI subsystem is referred to as the LTI mode of the LSS.

Example 1 *(Switching between sensors)*

Consider N LTI systems $P_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right]$ for $i = 1, 2, ..., N$. Suppose, due to certain restrictions on the communication channel, only output of one of these systems is measured and transmitted at each time step. In block

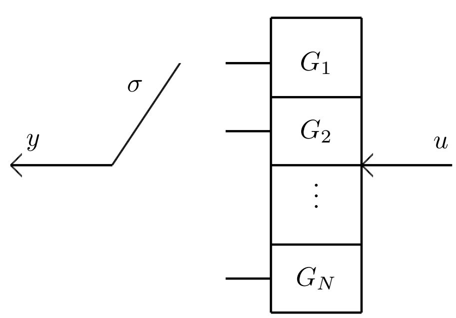


Figure 2.1: Output Switching System

diagram this system can be shown as in Figure 2.1. This system can be represented as

$$P_\sigma : \begin{cases} x(t+1) = \bar{A}x(t) + \bar{B}u(t) \\ y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{cases},$$

where $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_N$, $\bar{A} = diag(A_1, A_2, ..., A_N)$, *and* $\bar{B} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix}$.

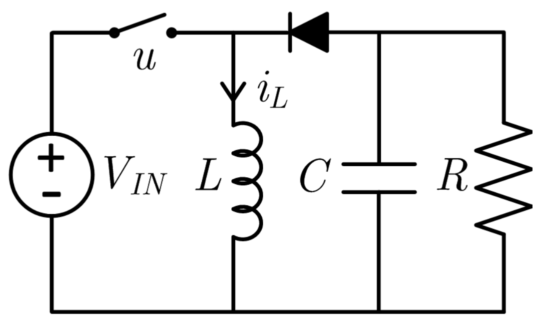


Figure 2.2: Buck-boost DC-DC converter

Example 2 *The buck-boost converter in Figure 2.2 can be mathematically written as*

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t),$$

where

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}, \\ B_0 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} \frac{V_{IN}}{L} \\ 0 \end{bmatrix}. \end{aligned}$$

The input-output properties of LSS is mostly studied in l_2 framework. In the context of l_1 or l_∞ induced gains, very little has been done. This is what we are concerned with in this part. In particular, we are concerned first with the worst-case l_∞ induced norm computation of LSS (Section 2.2). We consider a general LSS under the assumption

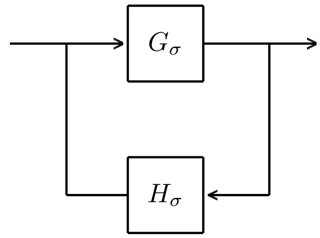


Figure 2.3: Interconnection of two switching systems

that the LSS is stable under arbitrary switching and study the following problem

$$\sup_{\sigma} \|G_\sigma\|, \tag{2.2}$$

where G_σ is a LSS, σ is the switching sequence, and the norm, $\|.\|$, is the l_∞ induced norm. This problem can be linked to robustness of systems when the switching is arbitrary. For example, invoking the small-gain theorem, the interconnection of two stable LSS G_σ and H_σ in Figure 2.3 is stable if $\sup_\sigma \|G_\sigma H_\sigma\| < 1$. We provide lower and upper bounds of the worst-case gain (2.2). These bounds are hard to compute and in general conservative. Hence, we narrow our attention to special classes of LSS by defining the classes of input, output, input-output, and the generalized input-output LSS and show that for these classes, exact and tractable expressions for the worst-case l_∞ induced norm can be found. The class of generalized input-output LSS proves important since any stable LSS can be approximated by one in this class with arbitrary accuracy. Moreover, we present a new necessary and sufficient condition equivalent to the stability of LSS in terms of a model matching problem that involves generalized input-output switching systems. Also, utilizing the generalized input-output switching systems we provide conditions to checking the gain of a LSS and also synthesize controllers for LSS via LP.

Moreover, in Section 2.3, we study the other extreme of (2.2) which is the minimal-gain problem. That is,

$$\inf_{\sigma} \|G_\sigma\|. \tag{2.3}$$

The interest in this type of problem is motivated by situations where there may be limited sensor and/or actuator resources for filtering and control. For example, there might be restrictions on how often a particular sensor or actuator is used. In these cases, the switching sequence may become an important decision variable to explore in minimizing estimation or tracking errors. More specifically, if the corresponding map of interest G_σ depends on a switching controller or filter Q_σ to be designed based on switching among a collection of LTI systems Q_i that depend on the availability of a sensor/actuator i, then a relevant performance optimization should involve both the selection of σ and Q_i. As one can see, studying the problem of minimizing the norm of a map of interest over the switching sequence σ is very relevant in order to tackle the bigger problem of optimizing jointly over the sequence σ and the controller/filter systems Q_i. This is so since, in principle, one can alternate between two disjoint optimizations over σ and Q_is to get a better solution at each time. In Section 2.3, we show that for FIR switching systems the minimizing switching sequence can be chosen to be periodic. For input-only or output-only switching systems an exact characterization of the minimal l_∞ gain is provided, and it is shown that the minimizing switching sequence is constant, which, as also shown, is not true for input-output switching.

2.1.1 Preliminaries and Notation

Here, we define the notation used in this part of the dissertation. First, notice that any linear causal map $T : u \in l_\infty \to y \in l_\infty$ can be thought of as an infinite dimensional lower triangular matrix,

$$T = \begin{bmatrix} T_{00} & 0 & 0 & \cdots \\ T_{11} & T_{10} & 0 & \cdots \\ T_{22} & T_{21} & T_{20} & \\ \vdots & \vdots & & \ddots \end{bmatrix}. \tag{2.4}$$

By $\mathcal{R}\left[T\right]_n$ we mean the causal part of the n^{th} block row in the matrix representation of T, i.e.

$$\mathcal{R}\left[T\right]_n := \begin{bmatrix} T_{nn} & T_{n,n-1} & \cdots & T_{n0} \end{bmatrix}.$$

In terms of this representation

$$\|T\| = \sup_n \left\|\mathcal{R}\left[T\right]_n\right\|.$$

We say a linear causal map $T : l_\infty \to l_\infty$, with matrix representation (2.4), is Finite Impulse Response (FIR) of some order $M \in \mathbb{Z}_+$ if for any integer $n \geq M$,

$$\mathcal{R}\left[T\right]_n = \begin{bmatrix} 0 & \cdots & 0 & T_{n,M-1} & \cdots & T_{n0} \end{bmatrix}.$$

A LTV system

$$G : \begin{cases} x\left(t+1\right) = A\left(t\right) x\left(t\right) + B\left(t\right) u\left(t\right) \\ y\left(t\right) = C\left(t\right) x\left(t\right) + D\left(t\right) u\left(t\right) \end{cases}, \text{ with } x\left(0\right) = x_0 \text{ given,}$$

where $u\left(t\right) \in \mathbb{R}^m, x\left(t\right) \in \mathbb{R}^n$, $y\left(t\right) \in \mathbb{R}^p$, and $x_0 \in \mathbb{R}^n$ are input, state, output, and the initial condition of the system and $A\left(t\right), B\left(t\right), C\left(t\right)$, and $D\left(t\right)$ are bounded matrices with appropriate dimensions for all $t \in \mathbb{Z}_+$, can be rewritten as

$$G : \begin{cases} \Lambda^{-1} x = \hat{A} x + \hat{B} u \\ y = \hat{C} x + \hat{D} u \end{cases}, \tag{2.5}$$

where $x = \{x(t)\}_{t=0}^{\infty}$, $y = \{y(t)\}_{t=0}^{\infty}$, $u = \{u(t)\}_{t=0}^{\infty}$, Λ is the delay operator,

$$\hat{A} = diag\left(A(0), A(1), ...\right) := \begin{bmatrix} A(0) & 0 & \cdots \\ 0 & A(1) & \\ \vdots & & \ddots \end{bmatrix},$$

and $\hat{B}$, $\hat{C}$, and $\hat{D}$ are defined analogously. We assume that $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are bounded maps. It can be easily shown that (2.5) can also be written

$$G : \begin{cases} x = \left(I - \Lambda\hat{A}\right)^{-1} \Lambda\hat{B}u + \left(I - \Lambda\hat{A}\right)^{-1} \bar{x}_0 \\ y = \hat{C}x + \hat{D}u \end{cases}, \tag{2.6}$$

where $\bar{x}_0 = \{x_0, 0, 0, ...\}$. In (2.6), the effects of the initial condition on the state variables are made explicit through the mapping $\left(I - \Lambda\hat{A}\right)^{-1}$.

Definition 3 *We say the LTV system G in (2.6) is stable if it is a bounded operator from* $\begin{pmatrix} x_0 \\ u \end{pmatrix}$ *to* $\begin{pmatrix} x \\ y \end{pmatrix}$.

We note that stability in the sense of the above definition is equivalent to the boundedness of the mapping $\left(I - \Lambda\hat{A}\right)^{-1}$ in (2.6). By a LSS we mean a system in the form (2.1). Notice that given $\sigma \in \Xi$, (2.1) defines a LTV system. Hence, one can define bounded linear operators $\hat{A}_\sigma$, $\hat{B}_\sigma$, $\hat{C}_\sigma$, and $\hat{D}_\sigma$ that depend on the switching sequence σ and rewrite (2.1) as

$$H_\sigma : \begin{cases} \Lambda^{-1}x = \hat{A}_\sigma x + \hat{B}_\sigma u \\ y = \hat{C}_\sigma x + \hat{D}_\sigma u \end{cases}, \tag{2.7}$$

where $\hat{A}_\sigma = diag\left(A_{\sigma_0}, A_{\sigma_1}, A_{\sigma_2}, ...\right)$ and $\hat{B}_\sigma$, $\hat{C}_\sigma$, and $\hat{D}_\sigma$ are defined analogously. Let $\mathcal{S}$ denote the class of LSS that are internally stable for any switching sequence. We use the compact notation of $H_\sigma = \left[\begin{array}{c|c} A_\sigma & B_\sigma \\ \hline C_\sigma & D_\sigma \end{array}\right]$ to denote (2.1) and by the i^{th} mode of H_σ, we mean the LTI system $\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right]$. We emphasize here that elements of $\mathcal{S}$ can be seen as maps from u to y that are l_∞-bounded uniformly with respect to the switching sequence [16]. That is, for $H_\sigma \in \mathcal{S}$, $\sup_\sigma \|H_\sigma\|$ is well defined.

2.2 Worst-Case l_∞ Induced Norm

In this section, we present our results regarding the computation of l_∞ induced norm of linear switched systems. More precisely, assuming that the LSS is internally stable for any switching sequence, we are interested in finding the the worst-case l_∞ norm of the system without imposing any constraint on the switching sequence. That is,

$$\sup_\sigma \|G_\sigma\| . \tag{2.8}$$

This proves to be an important problem since it links to the robustness with respect to switching. In general, the exact calculation of (2.8) is a highly complex problem. However, we can find bounds for (2.8). Towards this end, let

$$\begin{aligned} \alpha &: = \max_{j\in\mathbb{Z}_N} \|A_j\| , \beta := \max_{j\in\mathbb{Z}_N} \|B_j\| , \\ \gamma &: = \max_{j\in\mathbb{Z}_N} \|C_j\| , \theta = \max_{j\in\mathbb{Z}_N} \|D_j\| \end{aligned}$$

and let $\|G_j\|$ be the norm of the LTI system associated with the state space matrices of index j, i.e., $G_j = \left[\begin{array}{c|c} A_j & B_j \\ \hline C_j & D_j \end{array}\right]$. Furthermore, since we assume that G_σ is stable for any switching sequence, it is known that [16] for any matrix norm $\|.\|$, there exists some integer q such that

$$\rho := \max_{A_{i_k}\in\{A_j\}_{j\in\mathbb{Z}_N}} \left\|A_{i_1}A_{i_2}\dots A_{i_q}\right\| < 1. \tag{2.9}$$

The following proposition can be easily proved.

Proposition 4 *If $\alpha < 1$ then*

$$\max_j \|G_j\| \le \sup_\sigma \|G_\sigma\| \le \frac{\gamma\beta}{1-\alpha} + \theta.$$

If $\alpha \ge 1$ then

$$\max_j \|G_j\| \le \sup_\sigma \|G_\sigma\| \le \frac{\gamma\beta\bar{\alpha}}{1-\rho} + \theta,$$

where $\bar{\alpha} := 1 + \alpha + \alpha^2 + \cdots + \alpha^q$.

Proof. The lower bound follows trivially as the specific G_j correspond to a constant $\sigma(t) = j$ for all $t \ge 0$. For the case $\alpha < 1$ the result follows immediately as

$$\begin{aligned} \|y(t)\|_\infty &= \left\| C_{\sigma(t)} \sum_{\tau=0}^{t-1} A_{\sigma(t)} \dots A_{\sigma(t-\tau-1)} B_{\sigma(\tau)} u(\tau) + D_{\sigma(t)} u\,(t) \right\|_\infty \\ &\le \left[\gamma \sum_{\tau=0}^{t-1} \alpha^{t-\tau-1}\beta + \theta \right] \max_{\tau\le t} \|u(\tau)\|_\infty . \end{aligned}$$

The case $\alpha \ge 1$ follows in a similar pattern by bounding any product $A_{\sigma(t)} \dots A_{\sigma(t-\tau-1)}$ in chunks of size q as $\left\|A_{\sigma(t)} \dots A_{\sigma(t-\tau-1)}\right\| \le \rho^{\lfloor t/q \rfloor} \alpha^{t-\tau-1-\lfloor t/q \rfloor}$ if $\lfloor t/q \rfloor \le t-1$. ■

The above bounds can be in general conservative and, in the case where $\alpha \ge 1$, finding the integer q and hence ρ is a combinatorial problem, so these general bounds may not be very practical. This problem will be revisited later in

Subsection 2.2.4 where the general LSS are approximated by the generalized input-output switching systems. Also, it is obvious that if we define an average system $\bar{G} := \frac{1}{N}\sum_{j=1}^{N} G_j$ then $\|\bar{G}\| \leq \max_j \|G_j\|$ for any system norm. In the sequel we elaborate on specific classes of switched systems where exact expressions for $\sup_\sigma \|G_\sigma\|$ can be obtained. These are systems with non-switching state dynamics, that is they have a constant A-matrix. We begin with the output switching systems.

2.2.1 Output Switching Systems

The set of output switching systems, denoted by $\mathcal{S}_O^1$, is a subset of $\mathcal{S}$ whose elements, G_σ, can be realized as

$$G_\sigma : \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{cases},$$

where matrices A and B are constant and A is Schur stable. This class of systems can be thought of as the composition of a time-varying operator with a time-invariant one. More precisely, for

$$f = \left[\begin{bmatrix} f_1(0) \\ \vdots \\ f_N(0) \end{bmatrix}, \begin{bmatrix} f_1(1) \\ \vdots \\ f_N(1) \end{bmatrix}, \begin{bmatrix} f_1(2) \\ \vdots \\ f_N(2) \end{bmatrix}, \dots \right] \in l_\infty^{Np},$$

define the switching operator $S_\sigma : l_\infty^{Np} \to l_\infty^p$ as

$$(S_\sigma f)(t) = f_{\sigma(t)}(t), \text{ for } t \in \mathbb{Z}_+. \tag{2.10}$$

Using this operator any $G_\sigma \in \mathcal{S}_O$ can be written as

$$G_\sigma = S_\sigma \begin{bmatrix} G_1 \\ \vdots \\ G_N \end{bmatrix}, \tag{2.11}$$

where

$$G_i = \left[\begin{array}{c|c} A & B \\ \hline C_i & D_i \end{array} \right] \in \mathcal{L}_{TI}, \text{ for } i \in \mathbb{Z}_N.$$

Output switching systems have an obvious interpretation. As shown in Figure 2.4, they can be seen as N LTI systems driven by the same input and at each time step the output of only one of them is sampled. This, for example, can be the case when, due to some restrictions, only a subset of all sensors can provide measurements at each time step.

Given a switching sequence $\sigma = \{\sigma(0), \sigma(1), \dots\}$, G_σ defines a linear time varying operator whose infinite

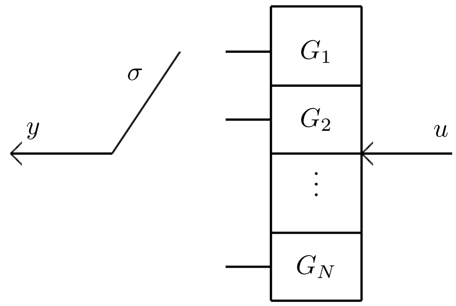


Figure 2.4: Output Switching System

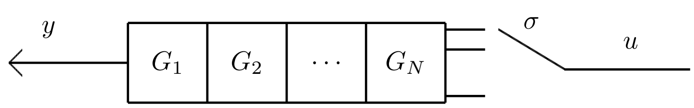


Figure 2.5: Input Switching System

dimensional lower triangular representation is as follows:

$$G_\sigma = \begin{bmatrix} g_{\sigma(0)}(0) & & & \\ g_{\sigma(1)}(1) & g_{\sigma(1)}(0) & & \\ g_{\sigma(2)}(2) & g_{\sigma(2)}(1) & g_{\sigma(2)}(0) & \\ \vdots & & & \ddots \end{bmatrix}, \tag{2.12}$$

where $\{g_i(k)\}_{k=0}^{\infty}$ is the impulse response of LTI system G_i. Clearly, the t^{th} row of (2.12), and consequently $y(t)$, depends only on the value of switching sequence at time t. In other words, the output of this system, y, at each time instant, t, is $y(t) = y_{\sigma(t)}(t)$ where $y_i = G_i u$, for $i \in \mathbb{Z}_N$. From the definition of l_∞ norm, one can write $\|y\|_\infty \leq \max_{i\in\mathbb{Z}_N} \|y_i\|_\infty$. Hence, based on the Proposition 4, it follows that $\sup_\sigma \|G_\sigma\| = \max_{i\in\mathbb{Z}_N} \|G_i\|$. Therefore, the following proposition is immediate.

Proposition 5 *For an output switching system of the form (2.11), we have*

$$\sup_\sigma \|G_\sigma\| = \left\| \begin{bmatrix} G_1 \\ \vdots \\ G_N \end{bmatrix} \right\|.$$

2.2.2 Input Switching Systems

Dual, in a sense, to the previous case is the input switching. The set of such systems is denoted by $\mathcal{S}_I$ and is a subset of $\mathcal{S}$ whose elements of the form

$$G_\sigma : \begin{cases} x(t+1) = Ax(t) + B_{\sigma(t)}u(t) \\ y(t) = Cx(t) + D_{\sigma(t)}u(t) \end{cases}.$$

In this case, the matrices A and C are constant, A is Schur stable, and the input matrices B and D switch. Such systems can be decomposed into an LTI system and a time varying operator. That is,

$$G_\sigma = \begin{bmatrix} G_1 & G_2 & \cdots & G_N \end{bmatrix} S_\sigma^*,$$

where

$$G_i = \left[\begin{array}{c|c} A & B_i \\ \hline C & D_i \end{array}\right] \in \mathcal{L}_{TI}, \text{ for } i \in \mathbb{Z}_N,$$

and S_σ^* is defined as

$$(S_\sigma^* u)(k) = \begin{bmatrix} 0 \\ \vdots \\ u(k) \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} \\ \\ \leftarrow \sigma(k)^{th} \text{ position} \\ \\ \\ \end{array}.$$

In block diagram, such systems are depicted in Figure 2.5. The norm computation of input switching systems is more involved than that of output switching ones. Hence, for the sake of clarity, we assume that there are only two modes, $N = 2$, and each mode is a single input single output system. We will relax these assumptions in Theorem 7.

We note that the infinite lower triangular representation of G_σ in this case is made of columns that belong to either G_1 or to G_2 depending on what $\sigma(t)$ is. It can be easily verified that the form of G_σ is as

$$G_\sigma = \begin{bmatrix} g_{\sigma(0)}(0) & & & \\ g_{\sigma(0)}(1) & g_{\sigma(1)}(0) & & \\ g_{\sigma(0)}(2) & g_{\sigma(1)}(1) & g_{\sigma(2)}(0) & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\{g_i(k)\}_{k=0}^{\infty}$ is the impulse response of G_i, for $i \in \{1, 2\}$. In this case, the norm of T^{th} row of matrix representation of G_σ is given by

$$\|\mathcal{R}[G_\sigma]_T\| = \sum_{k=0}^{T} \left|g_{\sigma(T-k)}(k)\right|.$$

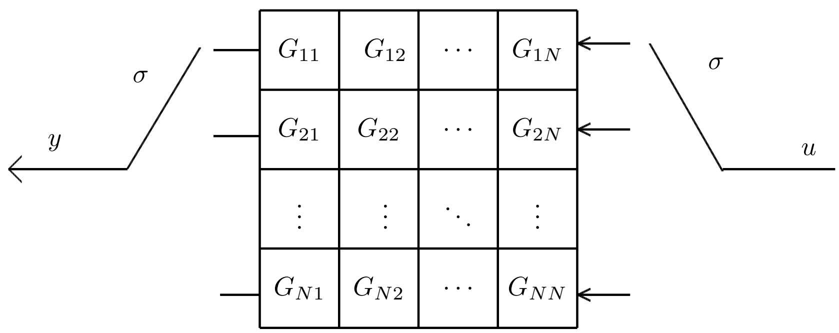


Figure 2.6: Input-Output Switching System

Clearly, the worst-case norm of the T^{th} row is obtained by

$$\max_{\sigma} \|\mathcal{R}\left[G_\sigma\right]_T\| = \sum_{k=0}^{T} \max\left\{\left|g_1\left(k\right)\right|, \left|g_2\left(k\right)\right|\right\},$$

where the use of $\max_\sigma$ is justified as the T^{th} row only has finitely many non-zero elements. Therefore, we have

$$\begin{aligned}\sup_{\sigma} \|G_\sigma\| &= \sup_{\sigma}\sup_{T} \|\mathcal{R}\left[G_\sigma\right]_T\| = \sup_{T}\max_{\sigma} \|\mathcal{R}\left[G_\sigma\right]_T\| \\ &= \sum_{k=0}^{\infty} \max\left\{\left|g_1\left(k\right)\right|, \left|g_2\left(k\right)\right|\right\}.\end{aligned}$$

Thus, the following is immediate:

Proposition 6 *Let* $G_\sigma = \begin{bmatrix} G_1 & G_2 \end{bmatrix} S_\sigma^*$ *be a SISO input switching system. Then, its worst-case* l_∞ *induced norm is given by*

$$\sup_{\sigma} \|G_\sigma\| = \|\bar{g}\|_1 = \sum_{t=0}^{\infty} |\bar{g}(t)|, \tag{2.13}$$

where $\bar{g} := \{\bar{g}(t)\}_{t=0}^{\infty} := \{\max\{|g_1(t)|, |g_2(t)|\}\}_{t=0}^{\infty}$.

Notice that for any $\varepsilon > 0$, since G_1 and G_2 are (exponentially) stable systems, one can apriori choose an integer n such that $\Pi^n \bar{g}$ is in any $\varepsilon > 0$ neighborhood of $\bar{g}$ in l_1 sense. Furthermore, generating $\bar{g}$ can be easily done as each term of $\bar{g}$ can be determined independently of the other terms by comparing only two numbers. Therefore, computing $\|\bar{g}\|_1$ to an apriori accuracy is a simple task.

2.2.3 Input-Output Switching Systems

Having defined input and output switching systems, it is intuitive to consider input-output switching systems. These are systems whose state matrix, A, remains constant and is Schur stable but the other matrices in their state space realization may switch among finitely many possibilities. The set of such systems is denoted by $\mathcal{S}_{IO}$ where each element G_σ can be realized as

$$G_\sigma : \begin{cases} x(t+1) = Ax(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{cases} . \tag{2.14}$$

Similar to input or output switching systems, these systems can be written as the composition of the switching operator with an LTI system as follows:

$$G_\sigma = S_\sigma \begin{bmatrix} G_{11} & \cdots & G_{1N} \\ \vdots & \ddots & \\ G_{N1} & \cdots & G_{NN} \end{bmatrix} S_\sigma^*, \tag{2.15}$$

where $G_{ij} = \left[\begin{array}{c|c} A & B_j \\ \hline C_i & D_i \end{array}\right] \in \mathcal{L}_{TI}$, for $i, j \in \mathbb{Z}_N$. Figure 2.6 is the block diagram of an input-output switching system. It is clear that this class of systems covers input only or output only switching systems. Here, we give an exact expression for computing the l_∞ induced norm of these (not necessarily SISO) systems. First, we state the results for Multi-Input Single-Output (MISO) systems and then argue how they can be generalized to the more general MIMO case.

Theorem 7 *For a MISO input-output switching system G_σ as in (2.15), the worst-case l_∞ induced norm can be calculated as*

$$\sup_\sigma \|G_\sigma\| = \max_{j\in\mathbb{Z}_N} \|\bar{g}_j\|_1 ,$$

where for each $j \in \mathbb{Z}_N$ the sequence $\bar{g}_j := \{\bar{g}_j(n)\}_{n=0}^\infty$ is defined as

$$\bar{g}_j(n) = \begin{cases} \|g_{jj}(0)\| & \text{for } n = 0 \\ \max_{k\in\mathbb{Z}_N} \|g_{jk}(n)\| & \text{for } n > 1 \end{cases} ,$$

and $\{g_{ij}(k)\}_{k=0}^\infty$ is the impulse response of G_{ij}.

Proof. It is easy to verify that the lower triangular infinite dimensional matrix representation of G_σ is given by

$$G_\sigma = \begin{bmatrix} g_{\sigma(0)\sigma(0)}(0) & & & \\ g_{\sigma(1)\sigma(0)}(1) & g_{\sigma(1)\sigma(1)}(0) & & \\ g_{\sigma(2)\sigma(0)}(2) & g_{\sigma(2)\sigma(1)}(1) & g_{\sigma(2)\sigma(2)}(0) & \\ \vdots & \vdots & & \ddots \end{bmatrix} .$$

The worst-case l_∞ induced norm, $\sup_\sigma \|G_\sigma\|$, for this operator is given by

$$\sup_t \sup_\sigma \left\| \left[g_{\sigma(t)\sigma(0)}(t) \;\; g_{\sigma(t)\sigma(1)}(t-1) \;\; \ldots \;\; g_{\sigma(t)\sigma(t)}(0) \right] \right\| .$$

Notice that the value of $\sigma(t) \in \mathbb{Z}_N$ has finite number of possibilities and does not affect the value of $\sigma(k)$ for $k < t$. Thus, we can write

$$\begin{aligned}
& \sup_{\sigma} \left\| \left[g_{\sigma(t)\sigma(0)}(t) \;\; g_{\sigma(t)\sigma(1)}(t-1) \;\; \ldots \;\; g_{\sigma(t)\sigma(t)}(0) \right] \right\| \\
= & \max_{j \in \mathbb{Z}_N} \sup_{\{\sigma(k)\}_{k=0}^{t-1}} \left\| \left[g_{j\sigma(0)}(t) \;\; g_{j\sigma(1)}(t-1) \;\; \ldots \;\; g_{jj}(0) \right] \right\| \\
= & \max_{j \in \mathbb{Z}_N} \sup_{\{\sigma(k)\}_{k=0}^{t-1}} \left(\sum_{\tau=0}^{t} \left\| g_{j\sigma(\tau)}(t-\tau) \right\| \right),
\end{aligned}$$

where we used the fact that G_σ is MISO. Therefore,

$$\begin{aligned}
\sup_{\sigma} \|G_\sigma\| &= \sup_{t} \max_{j \in \mathbb{Z}_N} \sup_{\{\sigma(k)\}_{k=0}^{t-1}} \left(\sum_{\tau=0}^{t} \left\| g_{j\sigma(\tau)}(t-\tau) \right\| \right) \\
&= \max_{j \in \mathbb{Z}_N} \lim_{t \to \infty} \sup_{\{\sigma(k)\}_{k=0}^{t-1}} \left(\sum_{\tau=0}^{t} \left\| g_{j\sigma(\tau)}(t-\tau) \right\| \right) = \max_{j \in \mathbb{Z}_N} \|\bar{g}_j\|_1,
\end{aligned}$$

and thus the proof is complete. ■

It is emphasized that, similar to the input switching case, determining $\bar{g}_j$ and the computation of its l_1 norm is tractable as each term in $\bar{g}_j$ can be determined independently of the other terms in $\bar{g}_j$. Furthermore, notice that the gain of an input-output switching system can be arbitrarily larger than that of its LTI modes. It can be done by having $\|g_{jk}(n)\|$, for $k \neq j$ and $n > 1$, larger than $\|g_{jj}(n)\|$ in the above theorem.

Now, suppose G_σ is a MIMO input-output switching system with m inputs and p outputs. That is, $y = G_\sigma u$, where $u = (u_1, u_2, ..., u_m) \in l_\infty^m$ and $y = (y_1, y_2, ..., y_p) \in l_\infty^p$. Define a MISO map H_σ^r to be the mapping from u to y_r, for $r = 1, 2, ..., p$. That is, G_σ can be partitioned as

$$G_\sigma = \begin{bmatrix} H_\sigma^1 \\ \vdots \\ H_\sigma^p \end{bmatrix},$$

where for any $r \in \{1, 2, ..., p\}$, H_σ^r is also an input-output switching system that can be written as

$$H_\sigma^r = S_\sigma \begin{bmatrix} H_{11}^r & \cdots & H_{1N}^r \\ \vdots & \ddots & \\ H_{N1}^r & \cdots & H_{NN}^r \end{bmatrix} S_\sigma^*,$$

with H_{ij}^r, for $i, j \in \mathbb{Z}_N$, can be realized as $H_{ij}^r = \left[\begin{array}{c|c} A & B_j \\ \hline \mathcal{R}[C_i]_r & \mathcal{R}[D_i]_r \end{array} \right]$. So, to compute the worst-case l_∞ norm of

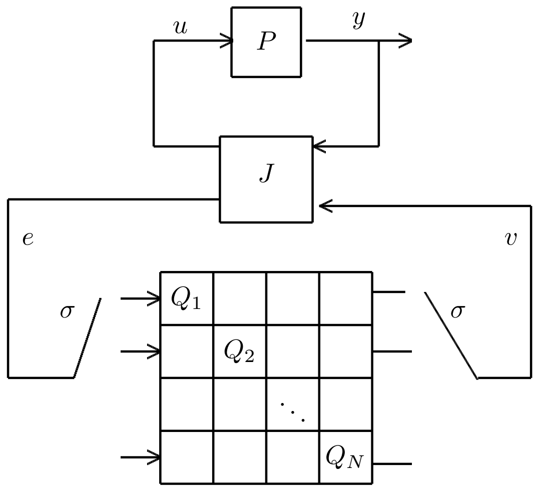


Figure 2.7: Switching Controller

G_σ we have:

$$\sup_\sigma \|G_\sigma\| = \sup_\sigma \frac{\|G_\sigma u\|_\infty}{\|u\|_\infty} = \sup_\sigma \max_{r\in\mathbb{Z}_p} \frac{\|H^r_\sigma u\|_\infty}{\|u\|_\infty} = \max_{r\in\mathbb{Z}_p} \sup_\sigma \|H^r_\sigma\|,$$

where $\sup_\sigma \|H^r_\sigma\|$ can be calculated exactly based on Theorem 7 as H^r_σ is MISO.

Remark 8 *The input-output switching systems are particularly important when, for a given LTI plant P, a finite set of stabilizing controllers, $\{K_i\}_{i=1}^N$, is given and one wants to realize them in a way that switching causes no instability. Let $\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I$ be a doubly-coprime factorization of P. Also, define $J := \begin{bmatrix} YX^{-1} & \tilde{X}^{-1} \\ X^{-1} & -X^{-1}N \end{bmatrix}$. Then, appealing to Youla-Kucera parameterization, each K_i can be written as the following lower linear fractional transformation $K_i = F_l(J, Q_i)$, see Figure 2.7. Clearly, switching between stabilizing controllers amounts to switching between Q_is. Furthermore, arbitrary switching between K_is results in a closed loop switching system which is stable if arbitrary switching between Q_is is stable. Therefore, consider an input-output switching system $Q_\sigma := S_\sigma \begin{bmatrix} Q_1 & & \\ & \ddots & \\ & & Q_N \end{bmatrix} S^*_\sigma$. Obviously, Q_σ is stable for arbitrary switching sequence and hence yields to a stabilizing switching controller $K_\sigma = F_l(J, Q_\sigma)$. Notice that K_σ has a more complicated structure than input-output switching system and it coincides with K_i for fixed switching sequence $\sigma \equiv i$. Similar ideas have been exploited in [52] and [21].*

Remark 9 *For the sake of gain computation, we mainly consider the initial condition of zero. However, there are problems for which one needs to guarantee robustness (in some sense) with respect to the bounded initial conditions as well as bounded inputs. In this case, consider (2.14) and without loss of generality suppose that the initial conditions and input satisfy $\|x_0\|_\infty \le 1$, $\|u\|_\infty \le 1$. We are interested to find*

$$\sup_\sigma \sup_{\|x_0\|_\infty \le 1, \|u\|_\infty \le 1} \|y\|_\infty .$$

Define $\tilde{u} := \begin{bmatrix} x_0 \\ u \end{bmatrix} \in l_\infty$ *and consider the mapping* $G_\sigma : \tilde{u} \to y$. *Clearly,* $\sup_\sigma \sup_{\|\tilde{u}\|_\infty \leq 1} \|y\|_\infty = \sup_\sigma \|G_\sigma\|$, *where* G_σ *has the following matrix representation,*

$$G_\sigma = \begin{bmatrix} C_{\sigma(0)} & g_{\sigma(0)\sigma(0)}(0) & & & \\ C_{\sigma(1)}A & g_{\sigma(1)\sigma(0)}(1) & g_{\sigma(1)\sigma(1)}(0) & & \\ C_{\sigma(2)}A^2 & g_{\sigma(2)\sigma(0)}(2) & g_{\sigma(2)\sigma(1)}(1) & g_{\sigma(2)\sigma(2)}(0) & \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}, \tag{2.16}$$

with $\{g_{ij}(n)\}_{n=0}^{\infty}$ *being the impulse response of* $G_{ij} = \left[\begin{array}{c|c} A & B_j \\ \hline C_i & D_i \end{array}\right]$. *Following the same line of argument as in Theorem 7, one can show the following:*

Theorem 10 *For a MISO input-output switching system with bounded non-zero initial condition (2.14),*

$$\sup_\sigma \sup_{\|\tilde{u}\|_\infty \leq 1} \|y\|_\infty = \max_{j \in \mathbb{Z}_N} \sup_M \left\|\bar{g}_j^M\right\|_1,$$

where for each $M \in \mathbb{Z}_+$ *and* $j \in \mathbb{Z}_N$, $\bar{g}_j^M := \left\{\bar{g}_j^M(n)\right\}_{n=0}^{M+1}$ *is defined by*

$$\bar{g}_j^M(n) = \begin{cases} \|g_{jj}(0)\| & \text{for } n = 0 \\ \max_{k \in \mathbb{Z}_N} \|g_{jk}(k)\| & \text{for } n \in \mathbb{Z}_M \\ \left\|C_j A^{M-1}\right\| & \text{for } n = M+1 \end{cases}.$$

This result can be extended to the MIMO case following the arguments proceeding Theorem 7. Furthermore, similar to the previous cases the computations to obtain the worst-case norm with arbitrary accuracy is tractable.

Next, we introduce the class of generalized input-output switching systems which can be used to approximate the worst-case gain of the general LSS (with A-matrix switching).

2.2.4 Approximation of LSS by Input-Output Switching Systems

Equation (2.1) represents a LSS in its generic form. Previously, we defined input-output switching systems and provided exact expression to compute their l_∞ gain. One can also extend the notion of input-output switching systems as follows:

Definition 11 *Let* M *be a positive integer. We say a LSS* P_σ *is an input-output LSS of degree* M *if it is stable and admits the realization*

$$P_\sigma : \begin{cases} x(t+1) = Ax(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\{\sigma(k)\}_{k=t-M+1}^{t}} x(t) + D_{\{\sigma(k)\}_{k=t-M+1}^{t}} u(t) \end{cases}. \tag{2.17}$$

The class of such systems is denoted by $\mathcal{S}_{IO}^{M}$ *and* $\mathcal{S}_{IO} = \bigcup_{M=1}^{\infty} \mathcal{S}_{IO}^{M}$.

We are also interested in a subclass of input-output LSS defined below:

Definition 12 *Let M be a positive integer. We say a LSS P_σ is an output-only LSS of degree M if it is stable and admits the realization*

$$P_\sigma : \begin{cases} x(t+1) = Ax(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\{\sigma(k)\}_{k=t-M+1}^{t}} x(t) + D_{\{\sigma(k)\}_{k=t-M+1}^{t}} u(t) \end{cases}.$$

The class of such systems is denoted by $\mathcal{S}_{O}^{M}$ *and* $\mathcal{S}_{O} = \bigcup_{M=1}^{\infty} \mathcal{S}_{O}^{M}$.

The classes of generalized input-output and output-only LSS are of particular interest for two reasons. First, any stable LSS can be approximated by elements of $\mathcal{S}_O$ and $\mathcal{S}_{IO}$ with arbitrary accuracy (see the next theorem). Second, we provide exact and tractable expressions to calculate the l_∞ induced norm of these systems.

Theorem 13 *Let G_σ be a stable LSS and $\varepsilon > 0$. Then, there exist an integer M, $\bar{G}_\sigma \in \mathcal{S}_{IO}^{M}$, and $\tilde{G}_\sigma \in \mathcal{S}_{O}^{M}$ such that*

$$\begin{aligned} \left\|G_\sigma - \bar{G}_\sigma\right\| &< \varepsilon, \\ \left\|G_\sigma - \tilde{G}_\sigma\right\| &< \varepsilon, \end{aligned}$$

for any switching sequence σ. Moreover, $\bar{G}_\sigma$ and $\tilde{G}_\sigma$ can be made FIR.

Proof. Let $G_\sigma = \left[\begin{array}{c|c} A_\sigma & B_\sigma \\ \hline C_\sigma & D_\sigma \end{array}\right]$ be a stable system. Suppose, $A_\sigma \in \mathbb{R}^{n\times n}$, $B_\sigma \in \mathbb{R}^{n\times m}$, $C_\sigma \in \mathbb{R}^{p\times n}$, $D_\sigma \in \mathbb{R}^{p\times m}$. Since, G_σ is stable, there exists and integer q such that for any integer $M \geq q$, $\max_{A_{i_k}\in\{A_j\}_{j\in\mathbb{Z}_N}} \|A_{i_1}A_{i_2}\dots A_{i_M}\| < \delta$. Let $i := q+2$ and define $\bar{G}_\sigma \in \mathcal{S}_{IO}^{i}$ with state-space matrices

$$\begin{aligned}
\bar{A} &= \begin{bmatrix} 0 & I_m & & \\ & 0 & \ddots & \\ & & \ddots & I_m \\ & & & 0 \end{bmatrix} \in \mathbb{R}^{(i-1)m\times(i-1)m}, \\
\bar{B}_{\sigma(t)} &= \begin{bmatrix} 0 & 0 & \cdots & B_{\sigma(t)} \end{bmatrix}^T \in \mathbb{R}^{(i-1)m\times m}, \\
\bar{C}_{\{\sigma(k)\}_{k=l}^{t}} &= C_{\sigma(t)} \begin{bmatrix} \prod_{k=l+1}^{t-1} A_k & \prod_{k=l+2}^{t-1} A_k & \cdots & I \end{bmatrix}, \\
\bar{D}_{\sigma(t)} &= D_{\sigma(t)},
\end{aligned}$$

where $l = \max\{t-i+1, 0\}$. It is easy to see for $t \geq i-1$, following the same argument as in the proof of Proposition

4,

$$\sup_{\sigma}\left\|G_{\sigma}-\bar{G}_{\sigma}\right\|\leq\max_{i}\|C_{i}\|\max_{i}\|B_{i}\|\frac{\delta}{1-\delta}.$$

Since one can choose δ such that $\max_i\|C_i\|\max_i\|B_i\|\frac{\delta}{1-\delta}<\varepsilon$, the proof is complete. ■

In the light of this theorem, to find the input-output gain of a generic LSS (with A-matrix switching), it suffices to find the worst-case gain of a generalized input-output switching system that is sufficiently close it. In the rest of this subsection, we show how to compute the l_∞ induced norm of a generalized input-output LSS P_σ of degree M. It is obvious from (2.17) that the C and D-matrices of P_σ can assume N^M values at each time instant t; each value associates with the segment $\{\sigma(t),\sigma(t-1),...,\sigma(t-M+1)\}$ of the switching sequence. Let $\mathcal{I}_M$ be the set of all Nvalued sequences of size M, i.e. $\mathcal{I}_M=\left\{i=\{i_k\}_{k=0}^{M-1}:i_k\in\mathbb{Z}_N\right\}$. We notice that each $P_\sigma\in\mathcal{S}_{IO}^M$ can be associated with N^{M+1} LTI systems denoted by $P_{i,j}$, where $i\in\mathcal{I}_M$, $j\in\mathbb{Z}_N$ and

$$P_{i,j}:\begin{cases}x(t+1)=Ax(t)+B_ju(t)\\ y(t)=C_ix(t)+D_iu(t)\end{cases}.$$

Then, the l_∞ gain of P_σ can be computed in terms of the impulse responses of the LTI systems $P_{i,j}$ denoted by $\{P_{i.j}(k)\}_{k=0}^{\infty}$, for $i\in\mathcal{I}_M$ and $j=\mathbb{Z}_N$.

Lemma 14 *Let P_σ be an input-output LSS of order M. Further, assume P_σ is multi-input and single-output (MISO). Then*

$$\sup_{\sigma}\|P_{\sigma}\|=\sup_{i=\{i_k\}_{k=0}^{M-1}\in\mathcal{I}_M}\sum_{k=0}^{M-1}\|P_{i,i_k}(k)\|+\sum_{k=M}^{\infty}\max_{j}\|P_{i,j}(k)\|\tag{2.18}$$

Remark 15 *Although, Lemma 14 addresses MISO systems, it can be easily extended to MIMO systems. In fact, suppose $P_\sigma:u\to\begin{pmatrix}y_1\\ \vdots\\ y_p\end{pmatrix}$ is a MIMO input-output LSS and let P_σ^k to be the MISO mapping from the input to the k^{th} output, i.e. $P_\sigma^k:u\to y_k$. Then, it can be verified that $\sup_\sigma\|P_\sigma\|=\max_k\sup_\sigma\left\|P_\sigma^k\right\|$ and Lemma 14 can be used to compute $\sup_\sigma\left\|P_\sigma^k\right\|$ as P_σ^k is MISO.*

Remark 16 *The l_∞ gain computation (2.18) can be written as a LP. For simplicity, suppose P_σ is SISO and it is FIR of order $T>M$ for all switching sequences. Then,*

$$\sup_{\sigma}\|P_{\sigma}\|=\min\gamma,$$

such that for any $i = \{i_k\}_{k=0}^{M-1} \in \mathcal{I}_M$, $k \in \{0, 1, ..., M-1\}$, $k' \in \{M, M+1, ..., T-1\}$, *and* $j \in \mathbb{Z}_N$

$$\begin{aligned} \left|P_{i,i_k}(k)\right| &\leq \gamma_i(k), \\ \left|P_{i,j}(k')\right| &\leq \gamma_i(k'), \\ \sum_{s=0}^{T-1} \gamma_i(s) &\leq \gamma. \end{aligned}$$

2.2.5 Stability of LSS and LTV Systems

Our perspective in this subsection is greatly influenced by the fact that LSS reduce to LTV systems for a fixed switching sequence. Hence, we take the approach of first establishing the results for LTV systems and then tailoring them to LSS. More precisely, we study the stability and stabilizability of LTV systems and reduce them to convex optimization problems. Then, we extend the results to LSS and argue how the stability/stabilizability problem can be converted to a partially nested sequence of LP.

Notice that the LTV system G in (2.6) is stable if and only if the mapping $\left(I - \Lambda\hat{A}\right)^{-1}$ is stable. In other words, G is stable if and only if $\hat{A}$ stabilizes Λ. Invoking the Youla-Kucera parameterization, $\hat{A}$ stabilizes Λ if and only if there exists a stable LTV system Q_A such that

$$\hat{A} = Q_A\left(I + \Lambda Q_A\right)^{-1} = \left(I + Q_A\Lambda\right)^{-1} Q_A,$$

or equivalently,

$$\begin{aligned} \hat{A}\left(I + \Lambda Q_A\right) - Q_A &= 0, && (2.19) \\ \left(I + Q_A\Lambda\right)\hat{A} - Q_A &= 0. && (2.20) \end{aligned}$$

Finding Q_A satisfying (2.19) or (2.20) and making them exact equalities is a computationally challenging task. However, as stability is a robust property, one can think of relaxing the above conditions while preserving the necessity and sufficiency of the results as follows:

Theorem 17 *Consider the LTV system G in (2.6). Then G is a stable if and only if there exists an LTV system Q such that one of the following equivalent conditions hold*

$$\begin{aligned} \left\|\hat{A}\left(I + \Lambda Q\right) - Q\right\| &< 1, && (2.21) \\ \left\|\left(I + Q\Lambda\right)\hat{A} - Q\right\| &< 1. && (2.22) \end{aligned}$$

Proof. First, suppose G is stable. Based on the argument preceding the theorem, $\hat{A}$ stabilizes Λ and hence there exists Q_A satisfying (2.19) and (2.20). Obviously, Q_A satisfies (2.21) and (2.22) as well.

Conversely, suppose (2.21) and (2.22) hold for some Q. Then the state equation of G can be written as

$$\begin{aligned} x - \bar{x}_0 &= \Lambda \hat{A} x + \Lambda B u \\ &= \Lambda Q \left(I + \Lambda Q\right)^{-1} x + \Lambda \left[\hat{A} - Q \left(I + \Lambda Q\right)^{-1}\right] x + \Lambda B u, \end{aligned}$$

where we added and subtracted $\Lambda Q \left(I + \Lambda Q\right)^{-1} x$ to the right hand side. Now, after moving this term to the left hand side, we obtain

$$\left[I - \Lambda Q \left(I + \Lambda Q\right)^{-1}\right] x = \Lambda \left[\hat{A} - Q \left(I + \Lambda Q\right)^{-1}\right] x + \Lambda B + \bar{x}_0.$$

Thus,

$$\begin{aligned} x &= \left[I - \Lambda Q \left(I + \Lambda Q\right)^{-1}\right]^{-1} \left\{\Lambda \left[\hat{A} - Q \left(I + \Lambda Q\right)^{-1}\right] x + \Lambda B + \bar{x}_0\right\} \\ &= \left(I + \Lambda Q\right) \Lambda \left[\hat{A} - Q \left(I + \Lambda Q\right)^{-1}\right] x + \left(I + \Lambda Q\right) \Lambda B u + \left(I + \Lambda Q\right) \bar{x}_0 \\ &= \Lambda \left[\left(I + Q\Lambda\right) \hat{A} - Q\right] x + \left(I + \Lambda Q\right) \Lambda B u + \left(I + \Lambda Q\right) \bar{x}_0. \end{aligned} \tag{2.23}$$

Using a small-gain like argument, one can show that x in (2.23) remains bounded for bounded u and x_0 if (2.22) holds. To prove (2.21), define $\eta = \left[\hat{A} - Q \left(I + \Lambda Q\right)^{-1}\right] x$. Then,

$$x = \left(I + \Lambda Q\right) \Lambda \eta + \left(I + \Lambda Q\right) \Lambda B u + \left(I + \Lambda Q\right) \bar{x}_0.$$

Clearly, x remains bounded if η does. We will show that the evolution of η is stable if (2.21) holds. It is easy to verify that

$$\eta = \left[\hat{A} \left(I + \Lambda Q\right) - Q\right] \Lambda \eta + \left[\hat{A} \left(I + \Lambda Q\right) - Q\right] \left(I + \Lambda Q\right) \Lambda B u + \left[\hat{A} \left(I + \Lambda Q\right) - Q\right] \bar{x}_0,$$

which is stable if (2.21) holds. ■

The above proof holds as long as the norms in (2.21) and (2.22) are induced norms from any vector space to the same vector space. Furthermore, (2.21) and (2.22) are convex. And indeed, for the l_∞ induced norm, we will show how (2.21) can be cast as a linear program. To this end, suppose Q is FIR of order T with the following matrix representation

$$Q = \begin{bmatrix} q_0(0) & & & & & \\ q_1(1) & q_1(0) & & & & \\ q_2(2) & q_2(1) & q_2(0) & & & \\ \vdots & & & \ddots & & \\ 0 & q_T(T-1) & q_T(T-2) & \cdots & q_T(0) & \\ & & & & & \ddots \end{bmatrix}$$

Then

$$\begin{aligned}
&\mathcal{R}\left[\hat{A}\left(I+\Lambda Q\right)-Q\right]_T \\
&= A_T \begin{bmatrix} q_{T-1}\left(T-1\right) & \cdots & q_{T-1}\left(0\right) & I \end{bmatrix} - \begin{bmatrix} 0 & q_T\left(T-1\right) & \cdots & q_T\left(0\right) \end{bmatrix},
\end{aligned}$$

and (2.21) can be written as

$$\begin{aligned}
&\left\|\hat{A}\left(I+\Lambda Q\right)-Q\right\| = \sup_t \left\|\mathcal{R}\left[\hat{A}\left(I+\Lambda Q\right)-Q\right]_t\right\| \\
&= \sup_{l,j}\left\{\left|e_l'\left[A_j-q_j\left(0\right)\right]\right| + \sum_{s=1}^{T-1}\left|e_l'\left[A_j q_{j-1}\left(s-1\right)-q_j\left(s\right)\right]\right| + \left|e_l^T\left[A_j q_{j-1}\left(T-1\right)\right]\right|\right\} < 1, \qquad (2.24)
\end{aligned}$$

where the absolute value $|.|$ is taken component wise and e_l is a vector of all zeros and one for the l^{th} entry. It is obvious, from (2.24), that finding Q is a linear program.

A LSS reduces to a LTV system for a given switching sequence σ. Hence, its stability can be checked via Theorem 17 for that particular switching sequence. Clearly, if we want to check the stability of an LSS for every switching sequence, we have the check the stability of every induced LTV system as stated below:

Corollary 18 *Consider the LSS P_σ as in (2.7). Let Ξ be a set of switching sequences containing the sequences of interest. Then, P_σ is stable for any $\sigma \in \Xi$ if and only if there exists a stable switching system Q_σ such that*

$$\sup_{\sigma\in\Xi}\left\|\hat{A}_\sigma\left(I+\Lambda Q_\sigma\right)-Q_\sigma\right\| < 1, \qquad (2.25)$$

$$\sup_{\sigma\in\Xi}\left\|\left(I+Q_\sigma\Lambda\right)\hat{A}_\sigma-Q_\sigma\right\| < 1. \qquad (2.26)$$

We note that conditions (2.25) and (2.26) are in the so-called model matching form. In what follows, we discuss how (2.25) can be cast as a linear program if the norm is the l_∞ induced. Notice that given Q_σ satisfying (2.25), it can be approximated arbitrarily closely by an FIR input-output or output-only LSS. Therefore, the following holds true:

Theorem 19 *Consider the LSS P_σ as in (2.7). Let Ξ be a set of switching sequences containing the sequences of interest. Then, P_σ is stable for any $\sigma \in \Xi$ if and only if there exists an integer M such that one of the following holds:*

$$\inf_{Q_\sigma\in\mathcal{S}_{IO}^M}\sup_{\sigma\in\Xi}\left\|\hat{A}_\sigma\left(I+\Lambda Q_\sigma\right)-Q_\sigma\right\| < 1, \qquad (2.27)$$

$$\inf_{Q_\sigma\in\mathcal{S}_{O}^M}\sup_{\sigma\in\Xi}\left\|\hat{A}_\sigma\left(I+\Lambda Q_\sigma\right)-Q_\sigma\right\| < 1. \qquad (2.28)$$

It is easy to see that for $Q_\sigma \in \mathcal{S}_{IO}^M$ or $Q_\sigma \in \mathcal{S}_{O}^M$, the mapping $\hat{A}_\sigma\left(I+\Lambda Q_\sigma\right)-Q_\sigma$ is indeed an input-output LSS of degree $M+1$. Therefore, Lemma 14 can be used to reduce (2.27) or (2.28) to LPs.

The following example illustrates the utility of our approach.

Example 20 *Consider a LSS with two modes,*

$$A_1 = \begin{bmatrix} 0.63 & -1 \\ -0.08 & 0.51 \end{bmatrix}, A_2 = \begin{bmatrix} -0.31 & 0.33 \\ -0.68 & 0.38 \end{bmatrix}.$$

It is interesting to note that there exists no common quadratic Lyapunov function for this system. However, the stability of this system is guaranteed by the above corollary since, using linear programming, it can be computed that

$$\inf_{Q_\sigma \in \mathcal{S}^1_{IO}} \sup_{\sigma} \left\| \hat{A}_\sigma \Lambda + \left(\hat{A}_\sigma \Lambda - I \right) Q_\sigma \Lambda \right\| = 0.7694 < 1.$$

2.2.6 Gain Computation for general LSS

In the previous section, we looked at the LTV systems (or LSS) as operators mapping $\begin{pmatrix} x_0 \\ u \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix}$ and derived conditions for their boundedness (stability). For a bounded operator, we will proceed to quantifying its bound, a.k.a. its gain. Conventionally, in the context of finding the gain of linear system, the initial condition is set to zero and further, without loss of generality, only the effect of u on y is studied. We emphasize that our computations can be cast as a LP. First, we state the results for LTV systems:

Theorem 21 *Consider the LTV system G in (2.6). Then G is stable and $\|G\| < 1$ if and only if there exist a stable LTV Q and $\delta > 0$ such that the following holds*

$$\left\| \begin{bmatrix} [A(I + \Lambda Q) - Q] & \delta [A(I + \Lambda Q) - Q] \Lambda B \\ \frac{1}{\delta} \hat{C} (I + \Lambda Q) & \hat{C} (I + \Lambda Q) \Lambda B + \hat{D} \end{bmatrix} \right\| < 1. \tag{2.29}$$

Proof. First, suppose G is stable and $\|G\| < 1$. That is, $\left\| \hat{C} \left(I - \Lambda \hat{A} \right)^{-1} \Lambda \hat{B} + \hat{D} \right\| < 1$ and $\hat{A}$ stabilizes Λ. Then, for $Q = \hat{A} \left(I - \Lambda \hat{A} \right)^{-1}$, (2.29) reads

$$\left\| \begin{bmatrix} 0 & 0 \\ \frac{1}{\delta} \hat{C} \left(I - \Lambda \hat{A} \right)^{-1} & \hat{C} (I + \Lambda Q) \Lambda B + \hat{D} \end{bmatrix} \right\| < 1,$$

which holds for sufficiently large δ.

Conversely, suppose (2.29) holds true. We want to show that G is stable and $\|G\| < 1$. The stability is guaranteed according to Theorem 17 as (2.29) implies $\|A(I + \Lambda Q) - Q\| < 1$. To prove $\|G\| < 1$, we use a small-gain theorem like argument. It is proved in [53] that if the interconnection of $G : \begin{pmatrix} x_0 \\ u \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix}$ and $\Delta : y \to u$ is stable for all Δ, possibly nonlinear and time-varying, with $\|\Delta\| \leq 1$, then $\|G\| < 1$. Therefore, it suffices to show that the condition in (2.29) guarantees the stability of the interconnection of G and Δ. Now, consider the expression of G as in (2.23). Define, $\eta = \left[A - Q\left(I + \Lambda Q\right)^{-1}\right] x$. Then, from (2.23), we have

$$x = \left(I + \Lambda Q\right) \Lambda \eta + \left(I + \Lambda Q\right) \Lambda B u + \left(I + \Lambda Q\right) \bar{x}_0.$$

From this, it is clear that the boundedness of η guarantees the boundedness of x and consequently the boundedness of y. Furthermore, one can write the evolution of η as

$$\begin{aligned} \eta &= \left[A - Q\left(I + \Lambda Q\right)^{-1}\right] x \\ &= \left[A\left(I + \Lambda Q\right) - Q\right] \Lambda \eta + \left[A\left(I + \Lambda Q\right) - Q\right] \Lambda B u + \left[A\left(I + \Lambda Q\right) - Q\right] \bar{x}_0. \end{aligned}$$

Now, instead of checking the stability of the interconnection of $G : \begin{pmatrix} x_0 \\ u \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix}$ and $\Delta : y \to u$, we check the stability of the interconnection of $H_1 : \begin{pmatrix} x_0 \\ \eta \\ u \end{pmatrix} \to \begin{pmatrix} \eta \\ y \end{pmatrix}$ and $H_2 : \begin{pmatrix} \eta \\ y \end{pmatrix} \to \begin{pmatrix} \eta \\ y \end{pmatrix}$ given by

$$H_1 : \begin{cases} \eta = \left[A\left(I + \Lambda Q\right) - Q\right] \Lambda \eta + \left[A\left(I + \Lambda Q\right) - Q\right] \Lambda B u + \left[A\left(I + \Lambda Q\right) - Q\right] \bar{x}_0 \\ y = \hat{C}\left(I + \Lambda Q\right) \Lambda \eta + \left[\hat{C}\left(I + \Lambda Q\right) \Lambda B + \hat{D}\right] u + \hat{C}\left(I + \Lambda Q\right) \bar{x}_0 \end{cases},$$

$$H_2 = \begin{bmatrix} I & 0 \\ 0 & \Delta \end{bmatrix}.$$

The interconnection of H_1 and H_2 is stable if

$$\left\| D^{-1} \begin{bmatrix} \left[A\left(I + \Lambda Q\right) - Q\right] & \left[A\left(I + \Lambda Q\right) - Q\right] \Lambda B \\ \frac{1}{\gamma}\hat{C}\left(I + \Lambda Q\right) & \frac{1}{\gamma}\hat{C}\left(I + \Lambda Q\right) \Lambda B + \hat{D} \end{bmatrix} D \right\| < 1,$$

for some D. In particular, for $D = \begin{bmatrix} I & 0 \\ 0 & \delta I \end{bmatrix}$ we have

$$\left\| \begin{bmatrix} \left[A\left(I + \Lambda Q\right) - Q\right] & \delta\left[A\left(I + \Lambda Q\right) - Q\right] \Lambda B \\ \frac{1}{\delta}\hat{C}\left(I + \Lambda Q\right) & \hat{C}\left(I + \Lambda Q\right) \Lambda B + \hat{D} \end{bmatrix} \right\| < 1.$$

■

We note that (2.29) is not convex in both δ and Q. It is, however, convex given δ. Condition (2.29) can be further simplified for the l_∞ case as follows:

Corollary 22 *The LTV system G is stable and $\|G\| < \gamma$ for some $\gamma > 0$ if and only if there exist a stable LTV Q and $\delta > 0$ such that the following hold*

$$\left\| \hat{A} \left(I + \Lambda Q \right) - Q \right\| + \delta \left\| \hat{A} \left(I + \Lambda Q \right) \Lambda \hat{B} - Q \Lambda \hat{B} \right\| < 1, \tag{2.30}$$

$$\frac{1}{\delta} \left\| \hat{C} \left(I + \Lambda Q \right) \right\| + \left\| \hat{C} \left(I + \Lambda Q \right) \Lambda \hat{B} + \hat{D} \right\| < \gamma. \tag{2.31}$$

We note that if G is stable, (2.21) holds true and hence for sufficiently small value of δ, (2.30) and (2.31) admit a solution (Q, γ). Therefore, in principle, one can start from small values δ and gradually increase δ until either (2.30) becomes infeasible or the desired performance level γ is met. In fact, if $\|G\| < \gamma$, then (2.30) and (2.31) admit a solution for large enough δ that is quantified in the next proposition.

Proposition 23 *Suppose $\|G\| < \gamma$. Then, the set of δ for which there exists a Q satisfying (2.30) and (2.31) contains the semi-infinite interval $(\delta_0, +\infty)$, where*

$$\delta_0 = \frac{\left\| \hat{C} \left(I - \Lambda \hat{A} \right)^{-1} \right\|}{\gamma - \|G\|}.$$

Proof. Notice that since $\|G\| < \gamma$, there exists Q_A such that (2.21) holds. For this Q_A, (2.30) is always satisfied and (2.31) reduces to

$$\frac{1}{\delta} \left\| \hat{C} \left(I - \Lambda \hat{A} \right)^{-1} \right\| + \|G\| < \gamma,$$

which clearly holds for all $\delta > \delta_0$. ∎

This proposition is particularly useful since it guarantees that if one keeps increasing δ and checking the feasibility of (2.30) and (2.31) for the given δ, the procedure eventually stops once δ is greater that δ_0. Theorem 21 can be extended for LSS as below:

Corollary 24 *Let LSS P_σ be given as in (2.7). Then P_σ is stable and $\|P_\sigma\| < \gamma$ for any $\sigma \in \Xi$ if and only if there exist $\delta, \gamma_i > 0$, for $i \in \{1, 2, 3, 4, \}$, a positive integer M, and $Q_\sigma \in \mathcal{S}_O^M$ such that*

$$\begin{aligned} \gamma_1 + \delta \gamma_2 \quad &< \quad 1, \\ \frac{1}{\delta} \gamma_2 + \gamma_3 \quad &< \quad \gamma, \end{aligned}$$

and

$$\sup_{\sigma \in \Xi} \left\| \hat{A}_\sigma (I + \Lambda Q_\sigma) - Q_\sigma \right\| < \gamma_1, \tag{2.32}$$

$$\sup_{\sigma \in \Xi} \left\| \hat{A}_\sigma (I + \Lambda Q_\sigma) \Lambda \hat{B}_\sigma - Q_\sigma \Lambda \hat{B}_\sigma \right\| < \gamma_2, \tag{2.33}$$

$$\sup_{\sigma \in \Xi} \left\| \hat{C}_\sigma (I + \Lambda Q_\sigma) \right\| < \gamma_3 \tag{2.34}$$

$$\sup_{\sigma \in \Xi} \left\| \hat{C}_\sigma (I + \Lambda Q_\sigma) \Lambda \hat{B}_\sigma + \hat{D}_\sigma \right\| < \gamma_4. \tag{2.35}$$

Using Lemma 14 and Remark 16, one can cast (2.32)-(2.35) as LPs.

2.2.7 Stabilizability

Consider a LTV system H with the exogenous input w, control input u, measured output y, and regulated output z

$$H : \begin{cases} \Lambda^{-1} x = \hat{A} x + \hat{B}^w w + \hat{B}^u u \\ z = \hat{C}^z x + \hat{D}^{zw} w + \hat{D}^{zu} u \\ y = \hat{C}^y x + \hat{D}^{yw} w \end{cases} . \tag{2.36}$$

It can be easily shown that a state-feedback controller $K : x \to u$ stabilizes the closed-loop if and only if $\hat{A} + \hat{B}K$ stabilizes Λ. According to Theorem 17, $\left[I - \Lambda \left(\hat{A} + \hat{B}K \right) \right]^{-1}$ is stable if and only if there exist two stable LTV systems Q and Z such that

$$\left\| \hat{A} (I + \Lambda Q) + \hat{B}^u Z - Q \right\| < 1, \tag{2.37}$$

where $Z = K (I + \Lambda Q)$. Clearly, (2.37) is convex in Q and Z and can be seen as a state-feedback stabilizability check for (2.36). We further develop an output-feedback stabilizability test as follows:

Theorem 25 *There exists a stabilizing output-feedback controller if and only if there exist stable LTV systems Q, Z^F, and Z^L such that*

$$\begin{aligned} \left\| \hat{A} (I + \Lambda Q) + \hat{B}^u Z^F - Q \right\| &< 1, \\ \left\| (I + Q\Lambda) \hat{A} + Z^L \hat{C}^y - Q \right\| &< 1. \end{aligned} \tag{2.38}$$

In this case the controller is given by

$$K : \begin{cases} \chi = \Lambda \left(\hat{A} + \hat{B}^u F + L \hat{C}^y \right) \chi - \Lambda L y \\ u = F \chi \end{cases} ,$$

where $F = Z^F (I + \Lambda Q)^{-1}$ and $L = (I + Q\Lambda)^{-1} Z^L$.

Proof. We notice that the controller K given above is analogous to the an observer-based controller for linear time invariant systems. In fact, the proof follows similarly to proving observer-based controllers stabilize LTI systems. This is done by defining new variable $\tilde{x} = \chi - x$ and showing that the evolution of $\begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$ is stable. In fact, the stability of $\begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$ is guaranteed as (2.38) implies $\hat{A} + \hat{B}^u F$ and $\hat{A} + L\hat{C}^y$ stabilize Λ. ■

It is obvious at this point that this theorem can be immediately extended to LSS by letting $Q \in \mathcal{S}_O^M$ and $Z^F, Z^L \in \mathcal{S}_O^{M+1}$. In this case, the mappings in (2.38) become input-output LSS of degree $M+1$ and (2.38) can be converted to a LP in the case of l_∞.

2.2.8 Control Synthesis

Based on our previous developments, one can synthesize controllers that guarantee certain performance level. Here, we discuss the state-feedback control synthesis.

Consider a LSS plant given by

$$P_\sigma : \begin{cases} \Lambda^{-1} x = \hat{A}_\sigma x + \hat{B}_\sigma^w w + \hat{B}_\sigma^u u \\ z = \hat{C}_\sigma^z x + \hat{D}_\sigma^{zw} w + \hat{D}_\sigma^{zu} u \end{cases},$$

and a switching state-feedback controller $K_\sigma : x \to u$. The closed-loop, Φ_σ, is given by

$$\Phi_\sigma : \Lambda^{-1} x = \hat{A}_\sigma^{cl} x + \hat{B}_\sigma^{cl} w, z = \hat{C}_\sigma^{cl} x + \hat{D}_\sigma^{cl} w,$$

where

$$\begin{aligned} \hat{A}_\sigma^{cl} &= \hat{A}_\sigma + \hat{B}_\sigma^u K, \hat{B}_\sigma^{cl} = \hat{B}_\sigma^w, \\ \hat{C}_\sigma^{cl} &= \hat{C}^z + \hat{D}^{zu} K, \hat{D}_\sigma^{cl} = \hat{D}_\sigma^{zw}. \end{aligned}$$

From Corollary 24 and letting $Z_\sigma = K_\sigma (I + \Lambda Q_\sigma)$, we have that Φ_σ is stable and $\|\Phi_\sigma\| < \gamma$ if and only if there exist $\delta, \gamma_i > 0$, for $i \in \{1, 2, 3, 4, \}$, a positive integer M, $Q_\sigma \in \mathcal{S}_O^M$, and $Z_\sigma \in \mathcal{S}_O^{M+1}$ (or $Z_\sigma \in \mathcal{S}_{IO}^{M+1}$) such that

$$\begin{aligned} \gamma_1 + \delta\gamma_2 &< 1, \\ \frac{1}{\delta}\gamma_2 + \gamma_3 &< \gamma, \end{aligned} \tag{2.39}$$

	Achievable l_∞ gain		
δ	$T=2$	$T=3$	$T=5$
1	17.71	15.02	11.72
5	4.71	4.42	3.97
10	3.11	2.80	2.63
50	2.01	1.72	1.49
100	2.01	1.68	1.41
1000	2.00	1.67	1.41
10000	2.00	1.67	1.41
	T : FIR order of Q_σ		

Table 2.1: Closed-loop gain

and

$$\sup_{\sigma\in\Xi}\left\|\hat{A}_\sigma\left(I+\Lambda Q_\sigma\right)+\hat{B}^u_\sigma Z_\sigma - Q_\sigma\right\| < \gamma_1, \tag{2.40}$$

$$\sup_{\sigma\in\Xi}\left\|\hat{A}_\sigma\left(I+\Lambda Q_\sigma\right)\Lambda\hat{B}^w_\sigma+\hat{B}^u_\sigma Z_\sigma\Lambda\hat{B}^w_\sigma - Q_\sigma\Lambda\hat{B}^w_\sigma\right\| < \gamma_2, \tag{2.41}$$

$$\sup_{\sigma\in\Xi}\left\|\hat{C}^z_\sigma\left(I+\Lambda Q_\sigma\right)+\hat{D}^{zu} Z_\sigma\right\| < \gamma_3 \tag{2.42}$$

$$\sup_{\sigma\in\Xi}\left\|\hat{C}^z_\sigma\left(I+\Lambda Q_\sigma\right)\Lambda\hat{B}^w_\sigma+\hat{D}^{zu} Z_\sigma\Lambda\hat{B}^w_\sigma + \hat{D}^{zw}_\sigma\right\| < \gamma_4. \tag{2.43}$$

We emphasize again that the mappings in (2.40)-(2.43) are input-output LSS and their l_∞ induced norm can be computed via LP.

Example 26 *Consider the barbell of length l illustrated in Figure 2.8. There is mass of size $m = 1kg$ that jumps from one end of the barbell to the other end. The actuator torque (control input) and the disturbance torque are labeled as u and τ, respectively. After letting l equal to the gravitational constant and discretizing the model at $2Hz$ we obtain $x\left(t+1\right) = A_{\sigma(t)}x\left(t\right) + B^\tau_{\sigma(t)}\tau_d\left(t\right) + B^u_{\sigma(t)}u\left(t\right)$ where*

$$\begin{aligned} A_1 &= \begin{bmatrix} 1.001 & 0.050 \\ 0.050 & 1.001 \end{bmatrix}, A_2 = \begin{bmatrix} 0.999 & 0.050 \\ -0.050 & 0.999 \end{bmatrix}, \\ B^\tau_1 &= B^\tau_2 = B^u_1 = B^u_2 = \begin{bmatrix} 0.001 \\ 0.050 \end{bmatrix}. \end{aligned}$$

We want to design a state-feedback controller, $K : x \to u$, that stabilizes the closed-loop and study the l_∞ gain of the closed-loop from the disturbance torque τ_d to the states and control input, i.e. $\begin{pmatrix} x \\ u \end{pmatrix}$. To this end, one can use (2.39) - (2.43). For this example, we let $Q_\sigma \in \mathcal{S}^1_O$ be FIR of some order T and $Z_\sigma \in \mathcal{S}^2_O$. Then we minimize (Q_σ, γ) subject to (2.39) - (2.43) for different values of δ as tabulated in Table 2.1.

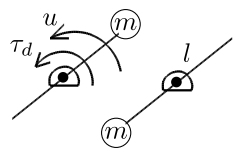


Figure 2.8: a barbell with switching mass at the end

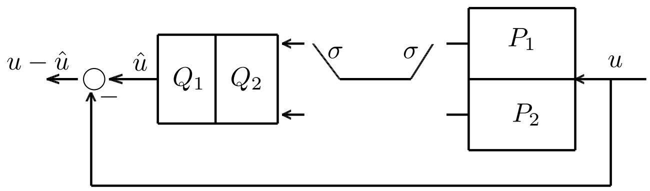


Figure 2.9: Filtering Problem

2.3 Minimal l_∞ Induced Norm

In the previous section, the problem of computing the worst-case l_∞ induced norm of switching systems over the switching sequence was considered. In this section, we consider the minimal gain problem $\inf_\sigma \|G_\sigma\|$. We show that an optimal switching sequence is constant, i.e. no need to switch, in case of output only or MISO input only switching G_σ and periodic in case of general switching FIR G_σ. As mentioned earlier, in some applications, the switching sequence may be used as a control variable. For example, one can consider a filtering problem depicted in Figure 2.9, where the interest is to estimate the input u by designing a filter $Q = (Q_1, Q_2)$ and a switching law to switch between different measurements. Having such a motivation, the following theorem may be used in characterizing minimizing switching sequences:

Theorem 27 *Let G_σ be a linear switched system as in (2.1). Suppose G_σ is FIR of order M for any σ. Then*

$$\inf_\sigma \|G_\sigma\| \tag{2.44}$$

is achieved by a periodic switching sequence with the period of at most N^M, where N is the number of switching modes of the system.

Proof. Suppose G_σ is FIR of order M. First, we will show that an optimal switching sequence exists. Let $\mathcal{R}[G_\sigma]_t$ denote the t^{th} block row of infinite dimensional lower triangular representation of G_σ. It is straight forward to verify that $\mathcal{R}[G_\sigma]_t$, for $t \geq M$, is uniquely determined by a segment of switching sequence of size M. More precisely, $\mathcal{R}[G_\sigma]_t$ is fully determined by the M-tuple $\{\sigma(n)\}_{n=t-M+1}^{t}$, if $t \geq M$ and by $\{\sigma(0), \sigma(1), ..., \sigma(t)\}$, if $t \leq M-1$. It is hence immediate that the entire set of switching sequences can generate at most N^M distinct rows in the matrix representation of the system. This makes (2.44) a finite dimensional optimization and thus an optimal switching sequence exists. Now let σ^* be an optimal switching sequence. We will show that one can create an optimal periodic switching sequence from σ^*. To this end define the set Ω_{σ^*} as $\Omega_{\sigma^*} := \left\{ (\sigma^*(k))_{k=t}^{t+M-1} : t \geq M \right\}$. That is Ω_{σ^*} is the set of M-tuples one can extract from the tail sequence $\{\sigma^*(t)\}_{t=M}^{\infty}$. As discussed above, Ω_{σ^*} can have at most N^M distinct elements. Thus, there should be at least one M-tuple that keeps showing up in $\{\sigma^*(t)\}_{t=M}^{\infty}$ infinitely often. That is there exists a sequence of time instants $\{t_k^*\}_{k=1}^{\infty}$ such that $\{\sigma^*(t)\}_{t=t_i^*}^{t_i^*+M-1} = \{\sigma^*(t)\}_{t=t_j^*}^{t_j^*+M-1}$ for $i, j \in \{1, 2, ...\}$. Pick i and j such that $t_j^* - t_i^* \geq M$. Define

$$\sigma_{per}(t) = \begin{cases} \sigma^*(t_i^* + M + t) & \text{if} \quad t \leq t_j^* + M - 1 \\ \sigma_{per}(t - M) & \text{if} \quad t \geq M \end{cases}.$$

Notice that, the rows generated by the switching sequence σ_{per}, i.e. $\mathcal{R}\left[G_{\sigma_{per}}\right]_t$ for $t \in \mathbb{Z}_+$, is a subset of those generated by σ^*. Therefore,

$$\left\|G_{\sigma_{per}}\right\| = \sup_t \left\|\mathcal{R}\left[G_{\sigma_{per}}\right]_t\right\| \leq \sup_t \left\|\mathcal{R}\left[G_{\sigma^*}\right]_t\right\| = \left\|G_{\sigma^*}\right\|.$$

This implies, $\left\|G_{\sigma_{per}}\right\| = \left\|G_{\sigma^*}\right\| = \inf_\sigma \left\|G_\sigma\right\|$. Furthermore, it is clear that the period is at most N^M. ■

The validity of the arguments in the proof of Theorem 27 strongly depends on the type of the norm. The fact that, in the l_∞ induced norm, one considers the rows of the matrix representation of the system is central. In particular, certain arguments in this proof fail if one tries to extend the results to the case of l_2 induced norm and considers the sub-matrices as opposed to the rows.

As mentioned earlier, if a switching system is stable for any switching sequence, for any matrix norm, there exists an integer q such that (2.9) holds. Hence, in general, any stable switching system in the form (2.1) can be approximated by an FIR system. Consequently, as a corollary of Theorem 27 we have:

Corollary 28 *For any $\varepsilon > 0$, there exists a periodic switching sequence σ^* such that*

$$\inf_\sigma \left\|G_\sigma\right\| - \varepsilon \leq \left\|G_{\sigma^*}\right\| \leq \inf_\sigma \left\|G_\sigma\right\| + \varepsilon.$$

Furthermore, there exists an FIR approximation of G_{σ^*}, denote it by $\bar{G}_{\sigma^*}$, such that

$$\inf_\sigma \left\|G_\sigma\right\| - \varepsilon < \left\|\bar{G}_{\sigma^*}\right\| < \inf_\sigma \left\|G_\sigma\right\|.$$

Proof. As G_σ is stable for any switching sequence σ, it can be uniformly approximated by FIR systems. That is, for $\varepsilon > 0$, there exists an FIR approximation $\bar{G}_\sigma$, of some order $M > 0$ such that

$$\left\|G_\sigma\right\| - \varepsilon < \left\|\bar{G}_\sigma\right\| < \left\|G_\sigma\right\|,$$

for any switching sequence σ. Therefore,

$$\inf_{\sigma} \|G_\sigma\| - \varepsilon < \inf_{\sigma} \left\|\bar{G}_\sigma\right\| < \inf_{\sigma} \|G_\sigma\| . \tag{2.45}$$

By Theorem 27, there exists a periodic sequence σ^* such that

$$\inf_{\sigma} \left\|\bar{G}_\sigma\right\| = \left\|\bar{G}_{\sigma^*}\right\| . \tag{2.46}$$

As, $\bar{G}_{\sigma^*}$ is the FIR approximation of G_{σ^*} we have

$$\|G_{\sigma^*}\| - \varepsilon < \left\|\bar{G}_{\sigma^*}\right\| < \|G_{\sigma^*}\| . \tag{2.47}$$

From (2.45),(2.46), and (2.47) we have

$$\inf_{\sigma} \|G_\sigma\| - \varepsilon \le \|G_{\sigma^*}\| \le \inf_{\sigma} \|G_\sigma\| + \varepsilon.$$

■

In the light of Theorem 27, one can consider a typical model matching problem

$$\inf_{(\sigma, Q_\sigma)} \|H_\sigma - U_\sigma Q_\sigma V_\sigma\| , \tag{2.48}$$

where H_σ, U_σ, V_σ, and Q_σ are FIR (and bounded) switching systems as in (2.1). Then, as an optimal switching σ exists that is periodic, finding an optimal Q_σ amounts to finding an optimal periodic solution to the above model matching problem.

Remark 29 *Notice that, one can consider the problem of (2.44) with added constraints on the switching sequence σ. Some of these constraints can be handled in the proof of Theorem 27 analogously. For example, if one requires that σ assumes all the values in the set $\mathbb{Z}_N$ infinitely many times, similar to the unconstraint problem, an optimal sequence will be periodic.*

Example 30 *As discussed above, even in the case of imposing certain constraints on the switching sequence, one can find an optimal periodic sequence. Consider the filtering problem depicted in Figure 2.9, where u is the input to the output switching channel, $P_\sigma := S_\sigma \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$, and $Q_\sigma := \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} S_\sigma^*$ is an input switching filter. At each time step, t, the measurement of $(P_\sigma u)(t)$ is fed to Q_σ. The interest is to estimate u by designing Q_1, Q_2, and the switching sequence σ. More precisely, the problem of interest is*

$$\inf_{Q_1, Q_2, \sigma} \|I - Q_\sigma P_\sigma\| .$$

We assume that there should be no channel that is used constantly for all time. This, for example, may represent an operational requirement. Therefore, the admissible switching requires that measurements from P_1 and P_2 are used infinitely many times, i.e., we exclude the cases that σ has a tail sequence identically equal to 1 *or* 2*. Mathematically, σ should satisfy the following:*

$$\nexists k \in \mathbb{Z}_+ : \Lambda^{-k}\sigma \text{ is constant.}$$

For this example, we assume P_1 and P_2 are FIR and their λ-transforms are given by

$$\hat{P}_1(\lambda) = -2 + 0.1\lambda, \hat{P}_2(\lambda) = -1.8 + 0.2\lambda.$$

Moreover, we search for Q_1 and Q_2 in the space of all FIR systems of order 2*. It is easy to verify that $I - Q_\sigma P_\sigma$ is FIR of order* 3*. Given σ, finding Q_1 and Q_2 is a convex problem. On the other hand, finding the minimizing σ is in general not convex. However, by Theorem 27 we know that σ is periodic and its basic period is at most $2^3 = 8$. Therefore, one can run an exhaustive search over the space of all possible switching sequence (which is finite but possibly large) to find and optimal σ. For this particular example, an optimal solution turns out to be*

$$\begin{aligned} \hat{Q}_1(\lambda) &= -0.4916 - 0.0556\lambda, \\ \hat{Q}_2(\lambda) &= -0.5556 - 0.0225\lambda, \\ \sigma &= (2, 1, 2, 1, 2, 1, ...), \end{aligned}$$

with optimal value $\inf_{Q_1,Q_2,\sigma} \|I - Q_\sigma P_\sigma\| = 0.0301$.

There are classes of systems for which a constant switching sequence (i.e. no switching) is the best strategy. It turns out that if G_σ is MISO input only or output only switching then the minimum norm can be achieved by a constant switching sequence. The result is given in what follows:

Theorem 31 *Let G_σ be an output only switching or a MISO input only switching system. Then*

$$\inf_\sigma \|G_\sigma\| = \min_{n\in\mathbb{Z}_N} \|G_n\|. \tag{2.49}$$

Proof. First suppose, G_σ is output switching. Then, given an input u and a switching sequence σ, the output $y := G_\sigma u$ at time t is given by

$$y(t) = C_{\sigma(t)} \sum_{k=0}^{t-1} A^{t-1-k} Bu(k) + D_{\sigma(t)} u(t).$$

Hence, for any switching sequence

$$\|y\|_\infty = \sup_t \|y(t)\|_\infty \geq \sup_t \min_{n\in\mathbb{Z}_N} \left\| C_n \sum_{k=0}^{t-1} A^{t-1-k} Bu(k) + D_n u(t) \right\|_\infty.$$

That is

$$\|G_\sigma\| \geq \min_{n\in\mathbb{Z}_N} \frac{\left\|C_n \sum_{k=0}^{t-1} A^{t-1-k}Bu(k) + D_n u(t)\right\|_\infty}{\|u\|_\infty} = \min_{n\in\mathbb{Z}_N} \|G_n\|,$$

where the equality is achieved for $\sigma(.) = \arg\min_{n\in\mathbb{Z}_N} \|G_n\|$.

Now, we will show (2.49) when G_σ is MISO input only switching. Without loss of generality, suppose there are only two modes of operation, i.e. the switching sequence takes values in the set $\{1,2\}$, and $\|G_1\| < \|G_2\|$. By the way of contradiction, suppose $\inf_\sigma \|G_\sigma\| < \|G_1\|$. Let

$$\varepsilon = \min\left\{\frac{\|G_1\| - \inf_\sigma \|G_\sigma\|}{2}, \frac{\|G_2\| - \|G_1\|}{2}\right\}.$$

Let $M > 0$ be the integer in the proof of Corollary 28 for such ε. Then, there exists a T-periodic switching sequence σ^*, for some $T > 0$, and an FIR system $\bar{G}_{\sigma^*}$ of order M, such that

$$\inf_\sigma \|G_\sigma\| - \varepsilon < \left\|\bar{G}_{\sigma^*}\right\| < \inf_\sigma \|G_\sigma\| \tag{2.50}$$

and

$$\|G_1\| - \varepsilon < \left\|\bar{G}_1\right\| < \|G_1\|, \tag{2.51}$$

where $\bar{G}_1$ is the FIR approximation of G_1 of order M. From (2.50) and (2.51) we have

$$\left\|\bar{G}_{\sigma^*}\right\| < \left\|\bar{G}_1\right\|. \tag{2.52}$$

Now, we will show if (2.52) holds, we arrive at the contradiction $\|G_2\| < \|G_1\|$, and hence $\inf_\sigma \|G_\sigma\| = \|G_1\|$. To this end, notice that, since $\bar{G}_{\sigma^*}$ and $\bar{G}_1$ are FIR of order M, $\left\|\bar{G}_1\right\| = \sum_{k=0}^{M-1} |G_1(k)|$ and $\bar{G}_{\sigma^*} = \sup_{t\geq M-1} \sum_{k=0}^{M-1} \left|G_{\sigma^*(t)}(k)\right|$. Hence, $\sum_{k=0}^{M-1} \left|G_{\sigma^*(t)}(k)\right| < \sum_{k=0}^{M-1} |G_1(k)|$, and

$$\sum_{t=M+1}^{M(T+1)} \sum_{k=0}^{M-1} \left|G_{\sigma^*(t)}(k)\right| < MT \sum_{k=0}^{M-1} |G_1(k)| = MT\left\|\bar{G}_1\right\|. \tag{2.53}$$

By changing the order of summation on the left hand side and direct calculation, one can verify

$$\sum_{t=M+1}^{M(T+1)} \sum_{k=0}^{M-1} \left|G_{\sigma^*(t)}(k)\right| = MT_1 \sum_{k=0}^{M-1} |G_1(k)| + MT_2 \sum_{k=0}^{M-1} |G_2(k)|, \tag{2.54}$$

where T_1 and T_2 are the number of times that G_1 or G_2 is active, respectively, in one period and $T_1 + T_2 = T$.

Therefore, from (2.53) and (2.54), we have

$$MT_1 \left\| \bar{G}_1 \right\| + MT_2 \left\| \bar{G}_2 \right\| < MT \left\| \bar{G}_1 \right\|,$$

or equivalently $\left\| \bar{G}_2 \right\| < \left\| \bar{G}_1 \right\|$. As

$$\begin{aligned} \|G_1\| - \varepsilon &< \left\| \bar{G}_1 \right\| < \|G_1\| \\ \|G_2\| - \varepsilon &< \left\| \bar{G}_2 \right\| < \|G_2\| \end{aligned}$$

and $\varepsilon \leq \frac{\|G_2\| - \|G_1\|}{2}$, we have

$$\|G_2\| - \varepsilon = \frac{\|G_2\| + \|G_1\|}{2} < \left\| \bar{G}_2 \right\| < \left\| \bar{G}_1 \right\| < \|G_1\|,$$

which in turn implies $\|G_2\| < \|G_1\|$. ■

Also, there are types of input-output switching systems (with fixed A matrix) for which a constant sequence is optimal. This is the case when one of the diagonal terms has the smallest norm as the following indicates.

Proposition 32 *For the MISO input-output switching system (2.15), if for some $i \in \mathbb{Z}_N$, $\|G_{ii}\| \leq \|G_{jk}\|$ for all $j, k \in \mathbb{Z}_N$, then*

$$\inf_{\sigma} \|G_\sigma\| = \|G_{ii}\|,$$

and the optimal sequence is the constant $\sigma(.) = i$.

Proof. For simplicity, we assume $G_\sigma = S_\sigma \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} S_\sigma^*$ with G_{ij} MISO for $i, j \in \{1, 2\}$. It is easy to see that

$$\inf_{\sigma} \|G_\sigma\| \geq \inf_{\sigma_1, \sigma_2} \left\| S_{\sigma_1} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} S_{\sigma_2}^* \right\|.$$

Notice that, $H_{\sigma_1 \sigma_2} := S_{\sigma_1} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} S_{\sigma_2}^*$ is an input-output switching system where the input matrices can switch independently of output matrices. By inspecting the rows of the matrix representation of $H_{\sigma_1 \sigma_2}$, we have

$$\inf_{\sigma_1, \sigma_2} \|H_{\sigma_1 \sigma_2}\| \geq \inf_{\sigma_2} \min \left\{ \begin{bmatrix} G_{11} & G_{12} \end{bmatrix} S_{\sigma_2}^*, \begin{bmatrix} G_{21} & G_{22} \end{bmatrix} S_{\sigma_2}^* \right\}.$$

Hence, if for some i, $\|G_{ii}\| \leq \|G_{ij}\|$ for $i, j \in \{1, 2\}$, according to Theorem 31, we have

$$\inf_{\sigma_2} \min \left\{ \begin{bmatrix} G_{11} & G_{12} \end{bmatrix} S_{\sigma_2}^*, \begin{bmatrix} G_{21} & G_{22} \end{bmatrix} S_{\sigma_2}^* \right\} = \|G_{ii}\|.$$

Clearly, this lower bound is achievable by G_σ and hence the proof is complete. ■

On the other hand, if the conditions of Proposition 32 are not satisfied, an optimal switching sequence may not necessarily be constant for an input-output LSS as the following example indicates.

Example 33 *Consider the (static) switched system (2.1) where the switching sequence takes values in the set* $\{1,2\}$ *and matrices* $A_i, B_i, C_i,$ *and* D_i *for* $i \in \{1,2\}$ *satisfy*

$$A_1 = A_2 = 0, C_i = B_i^T,$$

and

$$\begin{aligned} B_i^T B_i &= C_i B_i = 1, \\ B_1^T B_2 &= C_1 B_2 = C_2 B_1 = 0. \end{aligned}$$

Then, $y(t) = C_{\sigma(t)} B_{\sigma(t-1)} u(t-1)$. *It is easy to see, for constant* $\sigma(.)$, $y(t) = u(t-1)$. *However, for the periodic sequence* $\sigma = \{1,2,1,2,1,2,....\}$, $y(t) = 0$. *That is, the constant sequence is not optimal, while a periodic with period* 2 *is optimal.*

2.4 Miscellaneous Problems

In this section we provide some miscellaneous results on LSS. First, the composition of input and output switching systems is considered. We note that we can use input-output switching systems as building blocks to create more complicated structures. Then, the worst-case gain of slowly switching systems is studied. Finally, a sensitivity minimization and certain model matching problems are studied and it is shown that a LTV compensation cannot outperform a LTI one.

2.4.1 Composition of Output and Input Switching Systems

In some situations (e.g., see Section 2.3, Figure 2.9), one can have different compositions of input and output switching systems. Suppose $Q_\sigma \in \mathcal{S}_I$ and $P_\sigma \in \mathcal{S}_O$. In this case, it is of interest to study the worst-case norm of $P_\sigma Q_\sigma$ and $Q_\sigma P_\sigma$. It is easy to see that the former can be written as an input-output switching system and hence the previous results can be used for its norm computation. Indeed,

$$P_\sigma Q_\sigma = \left[\begin{array}{cc|c} A^q & 0 & B^q_\sigma \\ B^p C^q & A^p & B^p \\ \hline D^p_\sigma C^q & C^p_\sigma & D^p_\sigma D^q_\sigma \end{array}\right] \tag{2.55}$$

$$\begin{aligned} P_\sigma Q_\sigma &= S_\sigma \begin{bmatrix} P_1 \\ \vdots \\ P_N \end{bmatrix} \begin{bmatrix} Q_1 & \cdots & Q_N \end{bmatrix} S^*_\sigma \\ &= S_\sigma \begin{bmatrix} P_1 Q_1 & \cdots & P_1 Q_N \\ \vdots & \ddots & \\ P_N Q_1 & & P_N Q_N \end{bmatrix} S^*_\sigma, \end{aligned}$$

where $P_\sigma = \left[\begin{array}{c|c} A^p & B^p \\ \hline C^p_\sigma & D^p_\sigma \end{array}\right]$ and $Q_\sigma = \left[\begin{array}{c|c} A^q & B^q_\sigma \\ \hline C^q & D^q_\sigma \end{array}\right]$. Clearly, $P_\sigma Q_\sigma$ belongs to $\mathcal{S}_{IO}$ and Theorem 7 can be used to calculate its worst-case l_∞ induced norm.

On the other hand, the worst-case norm computation of $Q_\sigma P_\sigma$ is more involved as

$$Q_\sigma P_\sigma = \begin{bmatrix} Q_1 & \cdots & Q_N \end{bmatrix} S^*_\sigma S_\sigma \begin{bmatrix} P_1 \\ \vdots \\ P_N \end{bmatrix},$$

and is not in the form of an input-output switching system. In fact, consider the state-space realization of $Q_\sigma P_\sigma$

$$Q_\sigma P_\sigma = \left[\begin{array}{cc|c} A^p & 0 & B^p \\ B^q_\sigma C^p_\sigma & A^q & B^q_\sigma \\ \hline D^q_\sigma C^p_\sigma & C^q & D^q_\sigma D^p_\sigma \end{array}\right]. \tag{2.56}$$

Clearly, in (2.56), the state coefficient matrix is also switching. But note that this switching does not cause any instability if P_σ and Q_σ are stable (or equivalently A^q and A^p are Schur stable) . The next theorem provides lower and upper bounds for the worst-case norm of such systems. To this end, notice that infinite lower triangular representation of $S^*_\sigma S_\sigma$ only has diagonal terms. These diagonal terms can assume finitely many values. In fact, let

$$S^*_\sigma S_\sigma = \begin{bmatrix} s(0) & 0 & \cdots \\ 0 & s(1) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

For $k \in \mathbb{Z}_+$, $s(k)$ is a matrix with identity on the $\sigma(k)^{th}$ block row and column and zero anywhere else. Let the set of possible values of $s(k)$ be denoted by $\bar{S}$. For example, if the system only has two switching modes, i.e. $\sigma(k) = \{1, 2\}$, then

$$s(k) \in \bar{S} = \left\{ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right\}.$$

It is easy to see that there is a one to one corresponding between $\sigma(k)$ and $s(k)$. Moreover, as discussed before, Q_σ and P_σ can be decomposed into LTI systems and the switching operator (or its adjoint). Let,

$$Q_\sigma = QS_\sigma^*, \qquad P_\sigma = S_\sigma P,$$

where Q and P are LTI with impulse responses $\{q(k)\}_{k=0}^\infty$ and $\{p(k)\}_{k=0}^\infty$, respectively. Then, the following holds:

Proposition 34 *Let $Q_\sigma \in \mathcal{S}_I$ and $P_\sigma \in \mathcal{S}_O$. Suppose Q_σ is a multi-input single output system. Then*

$$\max_i \sup_T \left\| \sum_{k=0}^T h_T^i(k) \right\| \le \sup_\sigma \|Q_\sigma P_\sigma\| \le \sup_T \sum_{k=0}^T g_T(k). \tag{2.57}$$

where, for given $T \in \mathbb{Z}_+$, the finite sequences $g_T := \{g_T(k)\}_{k=0}^T$, $h_T^1 := \left\{h_T^1(k)\right\}_{k=0}^T$, and $h_T^2 := \left\{h_T^2(k)\right\}_{k=0}^T$ are given by

$$\begin{aligned} g_T(k) &= \max_{s\in\bar{S}} \sum_{\tau=0}^k \|q(T-k)\, sp(k-\tau)\|, \\ h_T^1(k) &= \max_{s\in\bar{S}} \sum_{\tau=0}^k q(T-k)\, sp(k-\tau), \\ h_T^2(k) &= \min_{s\in\bar{S}} \sum_{\tau=0}^k q(T-k)\, sp(k-\tau). \end{aligned}$$

Proof. Notice that, for $\tau, T \in \mathbb{Z}_+$ and $\tau \le T$, the entry at T^{th} block row and τ^{th} block column of lower triangular infinite dimensional representation of $Q_\sigma P_\sigma$ is given by $[Q_\sigma P_\sigma]_{T,\tau} = \sum_{k=\tau}^T q(T-k)\, s(k)\, p(k-\tau)$. Therefore,

$$\|Q_\sigma P_\sigma\| = \sup_T \sum_{\tau=0}^T \left\| \sum_{k=\tau}^T q(T-k)\, s(k)\, p(k-\tau) \right\|.$$

Clearly, $\|Q_\sigma P_\sigma\|$ can upper and lower bounded as:

$$\begin{aligned} \sup_T \left\| \sum_{k=0}^T \sum_{\tau=0}^k q(T-k)\, s(k)\, p(k-\tau) \right\| &\le \|Q_\sigma P_\sigma\| \\ \|Q_\sigma P_\sigma\| &\le \sup_T \sum_{k=0}^T \sum_{\tau=0}^k \|q(T-k)\, s(k)\, p(k-\tau)\|. \end{aligned} \tag{2.58}$$

Taking $\sup_\sigma$, we have:

$$\max_i \sup_T \left\| \sum_{k=0}^{T} h_T^i(k) \right\| \leq \sup_\sigma \|Q_\sigma P_\sigma\| \leq \sup_T \sum_{k=0}^{T} g_T(k),$$

which completes the proof. ■

We will show that the computations in this theorem are tractable. Noting that any stable system can be approximated by FIR systems with arbitrary accuracy, we will show how this theorem is computationally tractable when specialized to FIR systems. Hence, suppose Q_σ and P_σ (or equivalently Q and P) are FIR of some order M. Then, the non-zero part of sequences in Proposition 34 can be written as $g_T := \{g_T(k)\}_{k=\max(T-M,0)}^{T}$,$h_T^1 := \left\{h_T^1(k)\right\}_{k=\max(T-M,0)}^{T}$, and $h_T^2 := \left\{h_T^2(k)\right\}_{k=\max(T-M,0)}^{T}$ where

$$g_T(k) = \max_{s\in\bar{S}} \sum_{\tau=\max(k-M,0)}^{k} \|q(T-k)\, sp(k-\tau)\|, \tag{2.59}$$

$$h_T^1(k) = \max_{s\in\bar{S}} \sum_{\tau=\max(k-M,0)}^{k} q(T-k)\, sp(k-\tau), \tag{2.60}$$

$$h_T^2(k) = \min_{s\in\bar{S}} \sum_{\tau=\max(k-M,0)}^{k} q(T-k)\, sp(k-\tau). \tag{2.61}$$

It is easy to verify that for any integers $T_1, T_2 \geq 2M$,

$$g_{T_1} = g_{T_2}, h_{T_1}^1 = h_{T_2}^1, h_{T_1}^2 = h_{T_2}^2,$$

and (2.57) reduces to

$$\max_i \max_{T\in\mathbb{Z}_{2M}} \left\| \sum_{k=0}^{T} h_T^i(k) \right\| \leq \sup_\sigma \|Q_\sigma P_\sigma\| \leq \max_{T\in\mathbb{Z}_{2M}} \sum_{k=0}^{T} g_T(k). \tag{2.62}$$

Now, notice that, for given T, each element in the sequences g_T, h_T^1, and h_T^2 is determined independently of the other elements in these sequences. Furthermore, determining each element of these sequences, e.g., $g_T(k)$ for some T and k, amounts to calculating the summations in (2.59)-(2.61) N times (N is the number of switching modes and also the number of elements in $\bar{S}$) for each $s \in \bar{S}$ and letting $g_T(k)$ to be the maximum of these N summations. That is, given T, calculating (2.59)-(2.61) and consequently determining g_T, h_T^1, and h_T^2 is a simple task. Therefore the summations and the bounds in (2.62) are easily computable.

Although this theorem provides bounds that could be conservative in general, there are classes of systems for which these bounds are sharp. The following corollary can be easily verified.

Corollary 35 *Let $Q_\sigma = QS_\sigma^* \in \mathcal{S}_I$ and $P_\sigma = S_\sigma P \in \mathcal{S}_O$. Suppose Q_σ is a MISO system. Furthermore, suppose Q and P are positive systems, i.e. the infinite lower triangular representations of Q and P contain only non-negative* *terms. Then*

$$\sup_{\sigma} \|Q_\sigma P_\sigma\| = \sup_{T} \sum_{k=0}^{T} g_T(k). \tag{2.63}$$

Proof. The proof follows easily by noticing that in the proof of Theorem 34, if $Q_\sigma P_\sigma$ is a positive system, one has equality in (2.58). ■

Remark 36 *The above theorem holds true if instead of Q and P being positive, we assume $Q_\sigma P_\sigma$ is positive for a "suprimizing" switching sequence.*

The generalization of the results of this section to MIMO systems is immediate and follows the same line of argument proceeding Theorem 7.

2.4.2 Slowly Switching Systems

Motivated by [26], one can consider the l_∞ induced norm computation of a LSS in the case of slow switching. To this end, let $G_\sigma = \left[\begin{array}{c|c} A_\sigma & B_\sigma \\ \hline C_\sigma & D_\sigma \end{array}\right]$ be a given LSS. We define the set $S[\tau]$, for $\tau \in \mathbb{Z}_+$, to be a set of switching sequences for which any two consecutive switches is at least τ steps apart. Clearly, $\sup_{\sigma\in S[\tau]} \|G_\sigma\|$ is a non-increasing function of τ. Hence, the limit

$$\|G_\sigma\|_{ss} := \lim_{\tau\to\infty} \sup_{\sigma\in S[\tau]} \|G_\sigma\|$$

exists and we refer to it as the slowly switching gain of the system.

Proposition 37 *Given $G_\sigma = \left[\begin{array}{c|c} A_\sigma & B_\sigma \\ \hline C_\sigma & D_\sigma \end{array}\right]$ that switches between N modes, its slowly switching gain $\|G_\sigma\|_{ss}$ is given by*

$$\sup_{k} \max_{i,j} \left\| \begin{bmatrix} C_i A_i^k B_i & \dots & C_i A_i B_i & C_i B_i & D_i \\ C_j A_i^k B_i & \dots & C_j A_i B_i & C_j B_i & D_j \\ C_j A_j A_i^{k-1} B_i & \dots & C_j A_j B_i & C_j B_j & D_j \\ \vdots & & & & \\ C_j A_j^k B_i & & \dots & C_j B_j & D_j \end{bmatrix} \right\|. \tag{2.64}$$

Proof. For the sake of simplicity, we assume there are only two modes of switching and each mode is FIR of order M. That is, $A_1^k = A_2^k = 0$ for $k \geq M$. Also, as we are interested to characterize $\sup_{\sigma\in S[\tau]} \|G_\sigma\|$ as τ approaches infinity, assume $\tau > 2M$. Without loss of generality, suppose $\sigma(0) = 1$. Furthermore, suppose that the first switch occurs at the time instant $T \in \mathbb{Z}_+$. If this switching sequence belongs to $S[\tau]$ for some $\tau > 2M$, there exists and integers $k_1 > 2M$ such that $\sigma(T) = \sigma(T+1) = ... = \sigma(T+k_1) = 2$. Now, consider the t^{th} row of the lower triangular infinite dimensional representation of G_σ. If $1 \le t < T$,

$$\mathcal{R}[G_\sigma]_t = \begin{bmatrix} C_1 A_1^{t-2} B_1 & \dots & C_1 A_1 B_1 & C_1 B_1 & D_1 \end{bmatrix},$$

if $T \le t < T+M$,

$$\mathcal{R}[G_\sigma]_t = \Big[\; C_2 A_2^k A_1^T B_1 \quad C_2 A_2^k A_1^{T-1} B_1 \quad \dots \qquad (2.65)$$
$$\dots \quad C_2 A_2^k B_1 \quad C_2 A_2^{k-1} B_2 \quad \dots \quad C_2 B_2 \quad D_2 \; \Big],$$

and if $T+M \le t < T+k_1$,

$$\mathcal{R}[G_\sigma]_t = \Big[\; 0 \quad \dots \quad 0 \quad C_2 A_2^k B_1 \quad \dots$$
$$\dots \quad C_2 A_2^k B_1 \quad C_2 A_2^{k-1} B_2 \quad \dots \quad C_2 B_2 \quad D_2 \; \Big],$$

where $k = t - T$. Clearly, the effects of the first mode are not present on the rows $T+M$ to $T+k_1$ (or negligible if the modes are not FIR). Using the same rationale, it is easy to argue that finding the slowly switching gain of the system amounts to finding the worst-case norm of the LSS with over the space of switching sequences with maximum one switch. By the inspection of the matrix representation of the LSS with one switch, it is easy to see that the row with maximum l_1 norm is one of the rows in the matrix (2.64) and hence the proof is complete. ■

To show the tractability of this result, suppose that each mode of the LSS is FIR. More precisely, suppose there exists an integer M such that $A_i^M = 0$ for $i \in \mathbb{Z}_N$. Then, it is easy to see that to find the slowly switching gain of G_σ, one needs to evaluate (2.64) for $k = 2M$ and each pair of $(i, j) \in \mathbb{Z}_M \times \mathbb{Z}_M$. That is, the size of (2.64) grows linearly in the size of FIR and hence it is computationally tractable.

2.4.3 Sensitivity Minimization

Consider a sensitivity minimization problem as depicted in Figure 2.10. Suppose P_1 and P_2 are two stable systems and the output of the plant at each time instant is either the output of P_1 or P_2. In this case, the plant can be seen as an output switching plant $P_\sigma = S_\sigma \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$. The interest is to design a controller, K_σ, to minimize the map, Φ_σ, from output disturbances, d, to plant output, y, for the worst switching sequence. That is,

$$\inf_{K_\sigma} \sup_{\sigma} \|\Phi_\sigma(K_\sigma)\|.$$

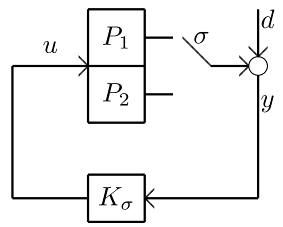


Figure 2.10: Sensitivity Minimization

Notice that the dependency of the controller on the switching signal is assumed in this problem. Since P_σ is stable, the set of all stabilizing controllers for this plant is parametrized by Youla-Kucera parameter as $K_\sigma = Q\left(I + P_\sigma Q\right)^{-1}$, where Q is any l_∞ bounded operator. It is well known that restricting Q to specific subsets of bounded operators, e.g. linear, nonlinear, time-invariant, or time-varying operators, spans different subsets of stabilizing controllers. For example, allowing Q to be linear switched system spans the set of stabilizing switched linear controllers. A class of tractable problems is obtained by restricting Q to be a linear input switching system. That is $Q_\sigma = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} S_\sigma^*$, where Q_1 and Q_2 are stable LTI systems. We remark here that these resulting controllers are a subset of all possible stabilizing controllers due to the fact that we prescribe the structure of Q as an input switching system. At this point, it is not clear how much is missed by imposing this structure on Q, but we certainly search over a large class of Ks which lead to exact convex optimization problems. Indeed, the resulting sensitivity map $\Phi_\sigma : d \mapsto y = (I - P_\sigma K_\sigma)^{-1}$ becomes

$$\Phi_\sigma = I + P_\sigma Q_\sigma. \tag{2.66}$$

A more general class of maps of this type, that includes (2.66) as special case and result in a convex optimization, is given by $\Phi_\sigma = H_\sigma + P_\sigma Q_\sigma$, where $H_\sigma = S_\sigma H S_\sigma^* \in \mathcal{S}_{IO}$, $P_\sigma = S_\sigma P \in \mathcal{S}_O$, $Q_\sigma = Q S_\sigma^* \in \mathcal{S}_I$ and H, P, and Q are LTI. Upon substitution of H_σ, P_σ, and Q_σ we obtain

$$\inf_{Q_\sigma} \sup_\sigma \|\Phi_\sigma\| = \inf_Q \sup_\sigma \|S_\sigma\left(H + PQ\right)S_\sigma^*\|, \tag{2.67}$$

which involves the minimization of the worst-case norm of an input-output switching system. Based on the development in the previous section (Theorem 7), minimizing $\|\Phi_\sigma\|$ over Q for the worst-case switching sequence is a convex problem.

Example 38 *Suppose* $H_\sigma = S_\sigma \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$, $P_\sigma = S_\sigma \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ *be output switching systems where*

$$\begin{aligned} \hat{H}_1(\lambda) &= -0.4 + 0.3\lambda - 0.2\lambda^2, \\ \hat{H}_2(\lambda) &= -0.1 + 0.2\lambda + 0.1\lambda^2, \\ \hat{P}_1(\lambda) &= 0.1 + 0.2\lambda, \hat{P}_2(\lambda) = -1 + 3\lambda. \end{aligned}$$

Notice that although H_σ is only output-switching but it can be written in a form consistent with (2.67) as $H_\sigma = S_\sigma \begin{bmatrix} H_1 & H_1 \\ H_2 & H_2 \end{bmatrix} S_\sigma^*$. *We want to design an input switching* $Q_\sigma = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} S_\sigma^*$ *to minimize the worst-case l_∞ induced norm of* $H_\sigma - P_\sigma Q_\sigma$. *This problem can be written as*

$$\begin{aligned} &\inf_{Q_\sigma} \sup_\sigma \|H_\sigma - P_\sigma Q_\sigma\| = \\ &\inf_{(Q_1,Q_2)} \sup_\sigma \left\| S_\sigma \begin{bmatrix} H_1 - P_1 Q_1 & H_1 - P_1 Q_2 \\ H_2 - P_2 Q_1 & H_2 - P_2 Q_2 \end{bmatrix} S_\sigma^* \right\|. \end{aligned} \tag{2.68}$$

As discussed before, this problem can be cast as a linear program which along the methods of [54], one can obtain the optimal value of 0.7386 *for*

$$\hat{Q}_1(\lambda) = 0.1684 - 0.0333\lambda, \hat{Q}_2(\lambda) = -0.1 - 0.0333\lambda.$$

In the next subsection, we will deal with the case of other model matching problems where Q_σ can be any linear switched system (not necessarily input switching) and we show under some conditions, if the plant is strictly causal, we can still reduce the problem to a tractable one.

2.4.4 Model Matching Problems

Recently in [55] a typical model matching problem was considered involving the output switching systems and the underlying norm being l_∞-induced or $\mathcal{H}_2$. The authors studied a problem of the form

$$\inf_{Q_{\sigma_Q}} \sup_{(\sigma_H,\sigma_U,\sigma_Q)} \left\| H_{\sigma_H} - U_{\sigma_U} Q_{\sigma_Q} \right\|,$$

where H_{σ_H}, U_{σ_U}, and Q_{σ_Q} are output switching systems associated respectively with (possibly different) switching sequences σ_H, σ_U, and σ_Q. It was shown that in the case of independent switching or partially matched switching, i.e. $\sigma_H = \sigma_U \neq \sigma_Q$, an output switching Q cannot out-perform an LTI Q.

In this section, we consider a similar problem and extend the results of [55] to show that a switched linear compensation of any type (not only output switching) cannot lead to a better performance over an LTI compensation if the compensator can switch independently of the plant or the plant is strictly causal.

To make the statements precise we have the following two theorems hold.

Theorem 39 *Let $H_{\sigma_H}, U_{\sigma_U}$, and Q_{σ_Q} belong to $\mathcal{S}$. If σ_Q switches independently of σ_U and σ_H, then*

$$\begin{aligned} \mu_0 :=& \inf_{Q\in\mathcal{S}} \sup_{(\sigma_H,\sigma_U,\sigma_Q)} \left\| H_{\sigma_H} - U_{\sigma_U} Q_{\sigma_Q} \right\| \\ =& \inf_{Z\in\mathcal{L}_{TI}} \sup_{(\sigma_H,\sigma_U)} \left\| H_{\sigma_H} - U_{\sigma_U} Z \right\|. \end{aligned} \tag{2.69}$$

Theorem 40 *Let σ be a switching sequence, $H_\sigma \in \mathcal{S}_O$, and $U_\sigma \in \mathcal{S}_O$ be output switching, and $Q_\sigma \in \mathcal{S}$ be any switching system. Further, assume U_σ is strictly causal. Then*

$$\nu_0 := \inf_{Q_\sigma \in \mathcal{S}} \sup_\sigma \|H_\sigma - U_\sigma Q_\sigma\| = \inf_{Z \in \mathcal{L}_{TI}} \sup_\sigma \|H_\sigma - U_\sigma Z\| . \tag{2.70}$$

Proof. Let $\varepsilon > 0$ be arbitrary. Then, there exists $Q_\sigma \in \mathcal{S}$ such that

$$\nu_0 \leq \sup_\sigma \|H_\sigma - U_\sigma Q_\sigma\| < \nu_0 + \varepsilon. \tag{2.71}$$

First, we show that for $\forall k \in \mathbb{Z}_+$,

$$\nu_k := \sup_\sigma \left\|H_\sigma - U_\sigma \Lambda^{-k} Q_\sigma \Lambda^k\right\| \leq \sup_\sigma \|H_\sigma - U_\sigma Q_\sigma\| . \tag{2.72}$$

To show this, since the associated norm is the l_∞ induced norm, for any $\varepsilon' > 0$, there exist a switching sequence σ' and $t' \geq 0$ such that

$$\nu_k - \varepsilon' < \left\|\mathcal{R}\left[H_{\sigma'} - U_{\sigma'} \Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'}\right\| \leq \nu_k.$$

Defined a sequence $\bar{\sigma}(.)$ as

$$\bar{\sigma}(t) = \begin{cases} \sigma'(t+k) & \text{for } t \neq t' \\ \sigma'(t') & \text{for } t = t' \end{cases} .$$

Then, one can write

$$\begin{aligned} & \mathcal{R}\left[H_{\sigma'} - U_{\sigma'} \Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'} \\ = \ & \mathcal{R}\left[H_{\sigma'}\right]_{t'} - \mathcal{R}\left[U_{\sigma'}\right]_{t'} \begin{bmatrix} \mathcal{M}\left[\Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'-1} \mathbf{0} \\ \mathcal{R}\left[\Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'} \end{bmatrix} \\ = \ & \mathcal{R}\left[H_{\bar{\sigma}}\right]_{t'} - \mathcal{R}\left[U_{\bar{\sigma}}\right]_{t'} \begin{bmatrix} \mathcal{M}\left[\Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'-1} \mathbf{0} \\ \mathcal{R}\left[\Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'} \end{bmatrix}, \end{aligned}$$

where $\mathbf{0}$ is a zero matrix with the same number of columns as $\mathcal{M}\left[\Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'-1}$. Notice that, $\mathcal{M}\left[\Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'-1} = \mathcal{M}\left[Q_{\bar{\sigma}}\right]_{t'-1}$, but $\mathcal{R}\left[\Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'} \neq \mathcal{R}\left[Q_{\bar{\sigma}}\right]_{t'}$ if $\sigma'(t'+k) \neq \sigma'(t')$. Also, notice that since $U_{\bar{\sigma}}$ is strictly causal the outcome of

$$\mathcal{R}\left[U_{\bar{\sigma}}\right]_{t'} \begin{bmatrix} \mathcal{M}\left[\Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'-1} \mathbf{0} \\ \mathcal{R}\left[\Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'} \end{bmatrix}$$

does not depend on $\mathcal{R}\left[\Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'}$ and hence one can write

$$\left\|\mathcal{R}\left[H_{\sigma'} - U_{\sigma'} \Lambda^{-k} Q_{\sigma'} \Lambda^k\right]_{t'}\right\| = \left\|\mathcal{R}\left[H_{\bar{\sigma}} - U_{\bar{\sigma}} Q_{\bar{\sigma}}\right]_{t'}\right\| .$$

Moreover

$$\left\|\mathcal{R}\left[H_{\bar{\sigma}}-U_{\bar{\sigma}}Q_{\bar{\sigma}}\right]_{t'}\right\| \leq \left\|H_{\bar{\sigma}}-U_{\bar{\sigma}}Q_{\bar{\sigma}}\right\| \leq \sup_{\sigma}\left\|H_{\sigma}-U_{\sigma}Q_{\sigma}\right\|.$$

Therefore,

$$\nu_k-\varepsilon' \leq \sup_{\sigma}\left\|H_{\sigma}-U_{\sigma}Q_{\sigma}\right\|,$$

for any $\varepsilon' > 0$ and this proves (2.72). Now, define the averaging system $Q_M^{\sigma} := \frac{1}{M+1}\left\{\sum_{k=0}^{M}\Lambda^{-k}Q_{\sigma}\Lambda^{k}\right\}$. Using (2.72) and the triangle inequality, it is easy to see $\sup_{\sigma}\left\|H_{\sigma}-U_{\sigma}Q_M^{\sigma}\right\| \leq \sup_{\sigma}\left\|H_{\sigma}-U_{\sigma}Q_{\sigma}\right\|$. Then, following the same line of argument as in [55, Theorem 3.1], [56], or [7], there exists a weak* convergent subsequence such that $Q_{LTI}^{\sigma} = \lim_{k\to\infty} weak^{*}\ Q_{M_k}^{\sigma}$, where $Q_{LTI}^{\sigma} \in \mathcal{L}_{TI} \subseteq \mathcal{S}$. Moreover,

$$\sup_{\sigma}\left\|H_{\sigma}-U_{\sigma}Q_{LTI}^{\sigma}\right\| \leq \sup_{\sigma}\left\|H_{\sigma}-U_{\sigma}Q_{\sigma}\right\|. \tag{2.73}$$

From (2.71) and (2.73) in one hand, and the fact that $Q_{LTI}^{\sigma_Q} \in \mathcal{S}$ on the other hand, one can write

$$\nu_0 \leq \sup_{(\sigma_H,\sigma_U)}\left\|H_{\sigma}-U_{\sigma}Q_{LTI}^{\sigma}\right\| \leq \nu_0+\varepsilon,$$

for every ε and this completes the proof. ■

In the light of these theorems, one can consider the problem in (2.68) of the previous section when H_{σ} and P_{σ} are output switching systems but Q_{σ} is a general linear switched system in $\mathcal{S}$. The following corollary is a straight forward consequence of Theorem 40 and Proposition 5.

Corollary 41 *Consider the map $\Phi := H_{\sigma_H}+P_{\sigma_P}Q_{\sigma_Q}$, where $H_{\sigma_H}, P_{\sigma_P} \in \mathcal{S}_O$ is output switching and $Q_{\sigma_Q} \in \mathcal{S}$. If the switching sequences σ_H, σ_P, and σ_Q are independent then*

$$\inf_{Q\in\mathcal{S}}\sup_{(\sigma_H,\sigma_P,\sigma_Q)}\|\Phi\| = \max_{i,j}\inf_{Z\in\mathcal{L}_{TI}}\left\|H_i+P_jZ\right\|. \tag{2.74}$$

Moreover, if $\sigma_H=\sigma_P=\sigma_Q$ and P_{σ_P} is strictly causal, then

$$\inf_{Q\in\mathcal{S}}\sup_{(\sigma_H,\sigma_P,\sigma_Q)}\|\Phi\| = \max_{i}\inf_{Z\in\mathcal{L}_{TI}}\left\|H_i+P_iZ\right\|. \tag{2.75}$$

We note here that (2.74) and (2.75) are standard l_1 problems and can be solved by standard methods in [54], [57], and [58].

Example 42 *Consider (2.70) with output switching H_{σ} and U_{σ} given as follows:*

$$\hat{H}_1(\lambda) = -0.4+0.3\lambda-0.2\lambda^2, \hat{H}_2(\lambda) = -0.1+0.2\lambda+0.1\lambda^2,$$

$$U_1 = \left[\begin{array}{c|c} \begin{bmatrix} 0.1 & 2 \\ -0.2 & -1 \end{bmatrix} & \begin{bmatrix} 0.3 \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} -1 & 0.5 \end{bmatrix} & 0 \end{array}\right],$$

$$U_2 = \left[\begin{array}{c|c} \begin{bmatrix} 0.7 & 0.2 \\ -1.8 & -0.3 \end{bmatrix} & \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \hline \begin{bmatrix} 0.2 & -0.3 \end{bmatrix} & 0 \end{array}\right].$$

In this case, Q_σ is any general LSS. Then, using the methods of [54], the problem has the optimal value of 0.6832 *and an optimal Q is LTI given by*

$$\hat{Q}\,(\lambda) = -0.0840 - 0.0756\lambda - 0.0252\lambda^2.$$

2.5 Summary

We presented results to characterize the worst (maximum) l_∞ gain of linear switched systems. It was shown that for certain classes of these LSS, namely for input-output switching systems, the exact computation of this gain is tractable and can be obtained via linear programming. Furthermore, the results on the input-output switching systems allow one to find tighter bounds for the gain of general switching systems. To this end, we introduced the richer class of the generalized input-output switching systems and showed that any stable LSS can be approximated by one in this class. Based on this class, we provided a new necessary and sufficient condition for the stability of LSS.

Moreover, it was shown that for general LSS, the computation of the gain is tractable when slowly switching is imposed. Certain control design optimization problems were studied for input-output LSS in the context of model matching and shown to be convex in the Youla-Kucera parameter. Further, in the same context and generalizing earlier works of the authors to general LSS, it was shown that switching compensators cannot out-perform LTI compensators in the case of unmatched switching sequences, or even in the case of matched switching when the plant is strictly causal.

Also in this chapter, we studied the problem of characterizing the minimum l_∞ gain of LSS over switching sequences. It was shown that for FIR systems, a minimizing sequence is periodic. The computation of its period however remains an open issue. For input only or output-only switching, it is shown that a constant switching sequence (i.e., no switching) is the minimizing one, which also readily determines the minimal l_∞ gain. For input-output switching on the other hand, periodic switching is in general necessary to minimize the l_∞ gain. All of these results hold true also when restrictions on the switching sequence (relating for example to percentage usage of each sensor or/and actuator) are imposed.

Chapter 3

Markov Linear Switched Systems

3.1 Introduction and Background

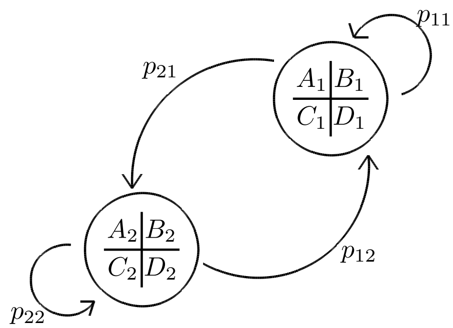
In the previous chapter, we studied the LSS with deterministic switching sequence; that is no statistics on the switching sequence are available. In this chapter, we consider LSS where switching is a random process. To this end, consider a LSS G_σ given by

$$G_\sigma : \begin{cases} x(t+1) = A_{\sigma(t)} x(t) + B^w_{\sigma(t)} w(t) + B^u_{\sigma(t)} u(t) \\ y(t) = C^y_{\sigma(t)} x(t) + D^{wy}_{\sigma(t)} w(t) \\ z(t) = C^z_{\sigma(t)} x(t) + D^{wz}_{\sigma(t)} w(t) + D^{uz}_{\sigma(t)} u(t) \end{cases}, \tag{3.1}$$

where $\sigma = \{\sigma(t)\}_{t=0}^{\infty}$ is the switching sequence taking finitely many values, w and u are the exogenous and control input, z and y are the regulated and measured output; matrices $A_{\sigma(t)}$, $B^\bullet_{\sigma(t)}$, $C^y_{\sigma(t)}$, $C^z_{\sigma(t)}$, $D^{\bullet z}_{\sigma(t)}$ and $D^{wy}_{\sigma(t)}$ are of appropriate dimension for $t \in \mathbb{Z}_+$, where $\bullet \in \{w, u\}$. We study the l_∞ performance and control synthesis of LSS in the form (3.1) when the switching sequence is a Markov process with a known transition matrix. We refer to such systems as Markovian Linear Switched Systems (MLSS). To study the l_∞-like performance of MLSS, we introduce a metric that mimics the l_∞ induced norm of a system in the deterministic framework. We call this metric the *stochastic l_∞ gain* of the system. This gain captures the maximal expected deviation of the output over inputs that could depend on the switching. As such, it can be used in situations where absolute value constraints are of interest e.g., position error in formation flight. We will present an exact expression to find the stochastic l_∞ gain. We will show that computing the stochastic l_∞ gain involves adding exponentially many terms and hence it is not easy to compute in general. As a trade-off, we will also consider the so-called mean performance of the MLSS and further synthesize an optimal control for minimizing the mean performance.

To formalize the notion of the stochastic l_∞ gain, we define the space of bounded random processes as

$$\mathcal{R}^n_\infty = \left\{ x = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (k) \right\}_{k=1}^{\infty} : \sup_k \mathbb{E}\left[\|x(k)\|_\infty \right] < \infty \right\},$$



where $\|x(k)\|_\infty = \max_{i\in\{1,2,\ldots,n\}}\{|x_i(k)|\}$ and $\mathbb{E}[.]$ stands for the expected value. The ball of bounded random processes can be defined as

$$\mathcal{BR}_\infty^n = \left\{x = \{x(k)\}_{k=1}^\infty \in \mathcal{R}_\infty^n : \sup_k \mathbb{E}\left[\|x(k)\|_\infty\right] \leq 1\right\}.$$

For a LSS

$$G_\sigma : \begin{cases} x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}^w w(t) \\ z(t) = C_{\sigma(t)}^z x(t) + D_{\sigma(t)}^{wz} w(t) \end{cases}, \tag{3.2}$$

where σ is a random process with known distribution, we define the stochastic l_∞ gain, from the exogenous input w to the regulated output z, as

$$\|G_\sigma\| := \sup_{\substack{w\in\mathcal{R}_\infty \\ \sup_k \mathbb{E}_{\sigma,w}\left[\|w(k)\|_\infty\right]\neq 0}} \frac{\sup_k \mathbb{E}_{\sigma,w}\left[\|z(k)\|_\infty\right]}{\sup_k \mathbb{E}_{\sigma,w}\left[\|w(k)\|_\infty\right]}. \tag{3.3}$$

We make sense of the above expression as follows: First, given a distribution for the random process w, z becomes a random process whose distribution depends on that of σ and w. Hence, the expectation is taken with respect to the distribution of σ and w in the numerator, i.e. $\mathbb{E}_{\sigma,w}\left[\|z(k)\|_\infty\right]$. Also, since w may depend on σ in general, the expectation in the denominator should be taken with respect to both σ and w, i.e. $\mathbb{E}_{\sigma,w}\left[\|w(k)\|_\infty\right]$. Therefore, for a given random process w the ratio $\frac{\sup_k \mathbb{E}_{\sigma,w}\left[\|z(k)\|_\infty\right]}{\sup_k \mathbb{E}_{\sigma,w}\left[\|w(k)\|_\infty\right]}$ is well-defined as long as $\sup_k \mathbb{E}_{\sigma,w}\left[|w(k)|\right] \neq 0$. Finally, we take the sup over all random processes w (possibly dependent on σ) with the property that $0 < \mathbb{E}_{\sigma,w}\left[\|w(k)\|_\infty\right] < \infty$. We will use $\mathbb{E}[.]$ instead of $\mathbb{E}_{\sigma,w}[.]$ when no confusion arises. Throughout this chapter, we make certain assumptions on the switching sequence and its dependency on the input as follow:

Assumption 43 *Given a nonnegative integer k, $\sigma(k+1)$ is conditionally independent of $\{w(t)\}_{t=0}^k$ given $\{\sigma(t)\}_{t=0}^k$.*

Assumption 44 *The switching sequence, σ, is a Markov process with the probability transition matrix $P = [p_{ji}]$. Furthermore, σ takes values in the set $\{1, 2\}$.*

Assumption 45 *The MLSS (3.1) is SISO. That is, the exogenous input and the regulated output are one dimensional.*

We emphasize that the Assumptions 44 and 45 are made, merely, for the simplicity of the presentation. The extension to the case when σ takes finitely many values or when the MLSS is MIMO is immediate. Furthermore, based on Assumptions 43 and 44, we have

$$\Pr\left(\sigma(k+1)=i|\sigma(k)=j,\{\sigma(t)\}_{t=0}^{k-1},\{w(t)\}_{t=0}^{k}\right)=p_{ij}.$$

As σ is a Markov process, we refer to the above LSS as *Markovian Linear Switched System* (MLSS). We use $\mathbf{1}$ to denote the standard indicator function. In particular, for $k\in\mathbb{Z}_+$, $\mathbf{1}_{\sigma(k)=i}$ is given by

$$\mathbf{1}_{\sigma(k)=i}=\begin{cases}1 \text{ if } \sigma(k)=i\\ 0 \text{ otherwise}\end{cases}.$$

3.2 Stochastic l_∞ Gain Calculation for MLSS

In this section, we compute the stochastic l_∞ gain of MLSS as defined in (3.3). To this end, consider the plant (3.2) with Assumptions 43, 44, and 45. The following theorem holds:

Theorem 46 *Consider the LSS in (3.2). Then, the stochastic l_∞ gain is given by*

$$\|G_\sigma\|=\max\{|D_1^{wz}|,[D_2^{wz}]\}+\sum_{k=0}^{\infty}\max\{S_1(k),S_2(k)\}, \tag{3.4}$$

where

$$S_i(k)=\sum_{i_1,\ldots,i_{k+1}}p_{i_{k+1}i_k}\ldots p_{i_1 i}\left|C_{i_{k+1}}^z\prod_{s=1}^{k}A_{i_s}B_i^w\right|, \tag{3.5}$$

for $\{i,i_1,i_2,\ldots,i_{k+1}\}\in\{1,2\}^{k+2}$.

Proof. Notice that from (3.2), for $k=1,2,\ldots$, we have

$$z(k)=\sum_{t=0}^{k-2}C_{\sigma(k)}^z\prod_{s=t+1}^{k-1}A_{\sigma(s)}B_{\sigma(t)}^w w(t)+C_{\sigma(k)}^z B_{\sigma(k-1)}w(k-1)+D_{\sigma(k)}^{wz}w(k). \tag{3.6}$$

Furthermore,

$$\sup_{w\in\mathcal{BR}_\infty}\mathbb{E}\left[|z(k+1)|\right]=\sup_{\substack{\mathbb{E}[|w(t)|]\leq 1\\ t=0,1,\ldots,k+1}}\mathbb{E}\left[|z(k+1)|\right]$$

$$\begin{aligned}
&= \sup_{\substack{\mathbb{E}[|w(t)|]\leq 1\\ t=0,1,\ldots,k+1}}\mathbb{E}\left[\left|\sum_{t=0}^{k-1}C^z_{\sigma(k+1)}\prod_{s=t+1}^{k}A_{\sigma(s)}B^w_{\sigma(t)}w(t)+C^z_{\sigma(k+1)}B_{\sigma(k)}w(k)+D^{wz}_{\sigma(k+1)}w(k+1)\right|\right]\\
&\leq \sup_{\mathbb{E}[|w(0)|]\leq 1}\mathbb{E}\left[\left|C^z_{\sigma(k+1)}\prod_{s=1}^{k}A_{\sigma(s)}B^w_{\sigma(0)}w(0)\right|\right]\\
&\quad+\sup_{\substack{\mathbb{E}[|w(t)|]\leq 1\\ t=1,1,\ldots,k+1}}\mathbb{E}\left[\left|\sum_{t=0}^{k-1}C^z_{\sigma(k+1)}\prod_{s=t+1}^{k}A_{\sigma(s)}B^w_{\sigma(t)}w(t)+C^z_{\sigma(k+1)}B_{\sigma(k)}w(k)+D^{wz}_{\sigma(k+1)}w(k+1)\right|\right]. \qquad (3.7)
\end{aligned}$$

It is easy to see that

$$\sup_{w\in\mathcal{BR}_\infty}\mathbb{E}\left[|z(k)|\right]=\sup_{\substack{\mathbb{E}[|w(t)|]\leq 1\\ t=1,1,\ldots,k+1}}\mathbb{E}\left[\left|\sum_{t=0}^{k-1}C^z_{\sigma(k+1)}\prod_{s=t+1}^{k}A_{\sigma(s)}B^w_{\sigma(t)}w(t)+C^z_{\sigma(k+1)}B_{\sigma(k)}w(k)+D^{wz}_{\sigma(k+1)}w(k+1)\right|\right].$$

Hence, from (3.7), we have

$$\sup_{w\in\mathcal{BR}_\infty}\mathbb{E}\left[|z(k+1)|\right]=S(k)+\sup_{w\in\mathcal{BR}_\infty}\mathbb{E}\left[|z(k)|\right], \qquad (3.8)$$

where

$$S(k)=\sup_{\mathbb{E}[|w(0)|]\leq 1}\mathbb{E}\left[\left|C^z_{\sigma(k+1)}\prod_{s=1}^{k}A_{\sigma(s)}B^w_{\sigma(0)}w(0)\right|\right].$$

In the last expression, $S(k)$ can be calculated as follows. Let $i_t\in\{1,2\}$, for $t=0,1,...,k+1$. Then,

$$\begin{aligned}
S(k) &= \sum_{i_0,i_1,\ldots,i_{k+1}}\sup_{\mathbb{E}[|w(0)|]\leq 1}\mathbb{E}\left[\left|C^z_{\sigma(k+1)}\prod_{s=1}^{k}A_{\sigma(s)}B^w_{\sigma(0)}w(0)\right|\mathbf{1}_{\sigma(0)=i_0}\ldots\mathbf{1}_{\sigma(k+1)=i_{k+1}}\right]\\
&= \sum_{i_0,i_1,\ldots,i_{k+1}}p_{i_{k+1}i_k}\ldots p_{i_1i_0}\sup_{\mathbb{E}[|w(0)|]\leq 1}\mathbb{E}\left[\left|C^z_{i_{k+1}}\prod_{s=1}^{k}A_{i_s}B^w_{i_0}w(0)\right|\mathbf{1}_{\sigma(0)=i_0}\right]\\
&= \max\left\{S_1(k),S_2(k)\right\},
\end{aligned}$$

where for $i\in\{1,2\}$

$$S_i(k)=\sum_{i_1,\ldots,i_{k+1}}p_{i_{k+1}i_k}\ldots p_{i_1i_0}\left|C^z_{i_{k+1}}\prod_{s=1}^{k}A_{i_s}B^w_i\right|.$$

Note that $\sup_{w\in\mathcal{BR}_\infty}\mathbb{E}\left[|z(0)|\right]=\max\left\{|D^z_1|,|D^z_2|\right\}$. Also, it is obvious from (3.8) that $\sup_{w\in\mathcal{BR}_\infty}\mathbb{E}\left[|z(k+1)|\right]$ is an increasing sequence and hence its sup happens when k approaches infinity. Taking the limit of (3.8) gives (3.4) and thus the proof is complete. ■

We would like to point out here that computing $S_i(k)$ in (3.5) involves adding up 2^{k+1} terms which grows exponentially with k. Therefore, the computation of the stochastic l_∞ gain for MLSS is in general harder than that of LTI systems. This computation, however, becomes easier for the case of input-output switching systems as is discussed next.

3.2.1 Input-Output Markov Linear Switched Systems

As mentioned in the earlier chapter of this dissertation, by input-output switching systems we mean those switching systems whose A-matrix remains constant. The next corollary is the direct consequence of Theorem 46.

Corollary 47 *For the input-output MLSS*

$$H_\sigma : \begin{cases} x(t+1) = Ax(t) + B_{\sigma(t)}w(t) \\ z(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}w(t) \end{cases}, \tag{3.9}$$

the stochastic l_∞ gain is given by

$$\|H_\sigma\| = \max\{|D_1|, |D_2|\} + \sum_{k=0}^{\infty} \max\{S_1(k), S_2(k)\}, \tag{3.10}$$

where for $\{i,j\} \in \{1,2\}^2$,

$$S_i(k) = \sum_j e_j^T \mathbb{P}^{k+1} e_i \left|C_j A^k B_i\right|, \tag{3.11}$$

and

$$\mathbb{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Proof. According to Theorem 46, when $A_1 = A_2 = A$, we have

$$\begin{aligned} S(k) &= \sum_{i_0, i_{k+1}} \sup_{\mathbb{E}[|w(0)|]\le 1} \mathbb{E}\left[\left|C_{i_{k+1}} \prod_{s=1}^{k} AB_{i_0} w(0)\right| \mathbf{1}_{\sigma(0)=i_0} \mathbf{1}_{\sigma(k+1)=i_{k+1}}\right] \\ &= \sum_{i_0, i_{k+1}} e_{i_{k+1}}^T \mathbb{P}^{k+1} e_{i_0} \sup_{\mathbb{E}[|w(0)|]\le 1} \mathbb{E}\left[\left|C_{i_{k+1}} \prod_{s=1}^{k} AB_{i_0} w(0)\right| \mathbf{1}_{\sigma(0)=i_0}\right] \\ &= \max\{S_1(k), S_2(k)\}. \end{aligned}$$

This together with Theorem 46 completes the proof. ■

We note here that the computations in (3.10) are tractable, in fact LP, and can be done with arbitrary accuracy. In the context of stochastic l_∞ gain, one could think of finding the minimal gain. That is, finding the probability distribution of the switching sequence such that the norm is minimized. This problem is considered and solved in the following theorem.

Theorem 48 *Consider the input-output MLSS in (3.9). Suppose,* $\Pr(\sigma(t+1) = j | \sigma(t) = i) = p_j$*. Then the minimal gain is given by the following LP.*

$$\inf_\sigma \|H_\sigma\| = \min_{\substack{\gamma, \gamma_0, \gamma_1, \ldots \\ p_1, p_2}} \gamma,$$

subject to

$$\begin{aligned}\gamma_0+\gamma_1+...+\gamma_k+... &\leq \gamma,\\ |D_i| &\leq \gamma_0,\\ p_1\left|C_1A^kB_i\right|+p_2\left|C_2A^kB_i\right| &\leq \gamma_k,\\ p_1+p_2 &= 1,\end{aligned}$$

for $i=1,2,$ *and* $k=1,2,3,....$

We emphasize here that the computations in the above theorem are LP and tractable with arbitrary accuracy.

3.3 Mean Performance

We studied the stochastic l_∞ gain of MLSS in the previous section. We argued that its computation is challenging as it involves adding exponentially many terms. In this section, we consider a different performance metric that can be computed easily. For the MLSS in (3.2), we define its mean performance as

$$\|G_\sigma\|_{MP}=\sup_{\substack{w\\ \sup_k|\mathbb{E}_{\sigma,w}[w(k)]|=1}}\sup_k\left|\mathbb{E}_{\sigma,w}\left[z\left(k\right)\right]\right|. \tag{3.12}$$

The rest of this section is devoted to characterizing the right hand side of (3.12). To this end, we will construct an LTI system that has $\mathbb{E}\left[z\left(k\right)\right]$ as its output. This system is induced from (3.1) and is given in the following proposition:

Proposition 49 *Consider the MLSS in (3.1). Then,*

$$\begin{aligned}&\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}(k+1)=\bar{A}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}(k)+\bar{B}^w\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}(k)+\bar{B}^u\begin{pmatrix}\upsilon_1\\ \upsilon_2\end{pmatrix}(k),\\ &\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}(k)=\bar{C}^y\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}(k)+\bar{D}^{wy}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}(k),\\ &\mathbb{E}\left[z\left(k\right)\right]=\begin{bmatrix}C_1^z & C_2^z\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}(k)+\bar{D}^{wz}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}(k)+\bar{D}^{uz}\begin{pmatrix}\upsilon_1\\ \upsilon_2\end{pmatrix}(k),\end{aligned} \tag{3.13}$$

where

$$\bar{A} = \begin{bmatrix} p_{11}A_1 & p_{12}A_2 \\ p_{21}A_1 & p_{22}A_2 \end{bmatrix},$$

$$\bar{B}^{\bullet} = \begin{bmatrix} \bar{B}_1^{\bullet} & \bar{B}_2^{\bullet} \end{bmatrix} = \begin{bmatrix} p_{11}B_1^{\bullet} & p_{12}B_2^{\bullet} \\ p_{21}B_1^{\bullet} & p_{22}B_2^{\bullet} \end{bmatrix},$$

$$\bar{C}^{y} = \begin{bmatrix} C_1^{y} & 0 \\ 0 & C_2^{y} \end{bmatrix}, \bar{D}^{wy} = \begin{bmatrix} D_1^{wy} & 0 \\ 0 & D_2^{wy} \end{bmatrix},$$

$$\bar{C}^{z} = \begin{bmatrix} C_1^{z} & C_2^{z} \end{bmatrix}, \bar{D}^{\bullet z} = \begin{bmatrix} D_1^{\bullet z} & D_2^{\bullet z} \end{bmatrix},$$

$$\eta_i(k) = \mathbb{E}\left[x(k)\mathbf{1}_{\sigma(k)=i}\right], \omega_i(k) = \mathbb{E}\left[w(k)\mathbf{1}_{\sigma(k)=i}\right],$$

$$\theta_i(k) = \mathbb{E}\left[y(k)\mathbf{1}_{\sigma(k)=i}\right], \upsilon_i(k) = \mathbb{E}\left[u(k)\mathbf{1}_{\sigma(k)=i}\right],$$

and $\bullet \in \{w, u\}$.

Proof. The proof follows similarly to that of Proposition 3.1 in [30] and hence is omitted here. ■

Definition 50 *We refer to the LTI representation (3.13) as the mean representation of* G_σ *and denote it by* $\mathbb{E}[G_\sigma]$.

We emphasize here that the proof of the above proposition depends on the validity of Assumption 43. It turns out that the mean performance of the system is completely characterized by its mean representation. Let $\bar{G}_i$ be the LTI mapping from ω_i to $\mathbb{E}[z(k)]$ with the impulse response $\{\bar{g}_i(k)\}_{k=0}^{\infty}$, for $i = 1, 2$. That is,

$$\bar{G}_i = \left[\begin{array}{c|c} \bar{A} & \bar{B}_i^{w} \\ \hline \bar{C}^{z} & D_i^{wz} \end{array}\right], \text{ for } i = 1, 2.$$

Then, the mean performance of G_σ is given in terms of $\bar{G}_i$ as stated in the next theorem.

Theorem 51 *Given a MLSS (3.1) with* $u = 0$, *its mean gain, from* w *to* z, *is given by*

$$\|G_\sigma\|_{MP} = \sum_{t=0}^{\infty} \max\{|\bar{g}_1(t)|, |\bar{g}_2(t)|\}.$$

Proof. By definition, we have that

$$\|G_\sigma\|_{MP} = \sup_{k} \sup_{\left|\sum_{i=1}^{2}\mathbb{E}\left[w(k)\mathbf{1}_{\sigma(k)=i}\right]\right|=1} |\mathbb{E}[z(k)]|. \tag{3.14}$$

Based on Proposition 49, $\mathbb{E}\left[z\left(k\right)\right]$, for given $k\in\mathbb{Z}_{+}$, reads

$$\mathbb{E}\left[z\left(k\right)\right]=\sum_{t=0}^{k}\bar{g}_{1}\left(k-t\right)\mathbb{E}\left[w\left(t\right)\mathbf{1}_{\sigma(t)=1}\right]+\sum_{t=0}^{k}\bar{g}_{2}\left(k-t\right)\mathbb{E}\left[w\left(t\right)\mathbf{1}_{\sigma(t)=2}\right]. \tag{3.15}$$

w^{*} is a maximizer and belongs to S if and only if it satisfies

$$\begin{pmatrix}\mathbb{E}\left[w^{*}\left(t\right)\mathbf{1}_{\sigma(t)=1}\right]\\ \mathbb{E}\left[w^{*}\left(t\right)\mathbf{1}_{\sigma(t)=2}\right]\end{pmatrix}=\begin{bmatrix}sgn\left(\bar{g}_{1}\left(k-t\right)\right)\\ 0\end{bmatrix},\text{ if }\left|\bar{g}_{1}\left(k-t\right)\right|\geq\left|\bar{g}_{2}\left(k-t\right)\right|, \tag{3.16}$$

$$\begin{pmatrix}\mathbb{E}\left[w^{*}\left(t\right)\mathbf{1}_{\sigma(t)=1}\right]\\ \mathbb{E}\left[w^{*}\left(t\right)\mathbf{1}_{\sigma(t)=2}\right]\end{pmatrix}=\begin{bmatrix}0\\ sgn\left(\bar{g}_{2}\left(k-t\right)\right)\end{bmatrix},\text{ if }\left|\bar{g}_{1}\left(k-t\right)\right|<\left|\bar{g}_{2}\left(k-t\right)\right|. \tag{3.17}$$

and hence

$$\sup_{w\in\mathcal{S}}\mathbb{E}\left[\left(G_{\sigma}w\right)\left(k\right)\right]=\sum_{t=0}^{k}\max\left\{\left|\bar{g}_{1}\left(k-t\right)\right|,\left|\bar{g}_{2}\left(k-t\right)\right|\right\}.$$

Taking $\sup_{k}$ from both sides results in

$$\left\|G_{\sigma}\right\|=\sup_{k}\sup_{w\in\mathcal{BR}_{\infty}}\mathbb{E}\left[\left(G_{\sigma}w\right)\left(k\right)\right]=\sum_{t=0}^{\infty}\max\left\{\left|\bar{g}_{1}\left(t\right)\right|,\left|\bar{g}_{2}\left(t\right)\right|\right\}.$$

■

In what follows, we address the control synthesis with respect to the mean performance and show how this problem is analogous to control synthesis for an LTI system with added constraints on the D-matrix of the controller.

3.3.1 Control Synthesis

Here, we are interested in designing a controller to stabilize and minimize the mean performance of the system from the exogenous input to regulated output. The stability in this section is taken with respect to the mean performance. That is, a stable system is the one with bounded mean output for bounded mean input. We consider a mode-dependent linear switched controller K_{σ} in the form

$$K_{\sigma}:\begin{cases}x_{C}\left(t+1\right)=A_{\sigma(t)}^{C}x_{C}\left(t\right)+B_{\sigma(t)}^{C}y\left(t\right)\\ u\left(t\right)=C_{\sigma(t)}^{C}x_{C}\left(t\right)+D_{\sigma(t)}^{C}y\left(t\right)\end{cases}. \tag{3.18}$$

The interconnection of (3.1) and (3.18) is denoted by $T\left(G_{\sigma},K_{\sigma}\right)$. This is the closed loop system mapping w to z. The control synthesis problem amounts to

$$\inf_{K_{\sigma}\text{ stabilizing}}\left\|T\left(G_{\sigma},K_{\sigma}\right)\right\|_{MP}. \tag{3.19}$$

Similarly to the mean representation of the plant, one can write a mean representation for K_σ and $T(G_\sigma, K_\sigma)$ as LTI systems mapping $\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ to $\begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix}$ and $\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ to $\mathbb{E}[z(k)]$, respectively. These LTI systems are denoted by $\mathbb{E}[K_\sigma]$ and $\mathbb{E}[T(G_\sigma, K_\sigma)]$. It is straight forward to verify that

$$\mathbb{E}[T(G_\sigma, K_\sigma)] = T(\mathbb{E}[G_\sigma], \mathbb{E}[K_\sigma]), \tag{3.20}$$

where $T(\mathbb{E}[G_\sigma], \mathbb{E}[K_\sigma])$ is the interconnection of the two LTI mean representations $\mathbb{E}[G_\sigma]$ and $\mathbb{E}[K_\sigma]$. Notice that $\mathbb{E}[T(G_\sigma, K_\sigma)]$ can be partitioned as

$$\mathbb{E}[T(G_\sigma, K_\sigma)] = T(\mathbb{E}[G_\sigma], \mathbb{E}[K_\sigma]) = \begin{bmatrix} T_1 & T_2 \end{bmatrix},$$

where T_i is an LTI system mapping ω_i to $\mathbb{E}[z(k)]$, for $i = 1, 2$. It is clear from (3.20) that, given K_σ, T_i depends on $\mathbb{E}[K_\sigma]$. Let $\left\{\bar{t}_i^{\mathbb{E}[K_\sigma]}(k)\right\}_{k=0}^{\infty}$ be the impulse response of T_i, where the dependence on $\mathbb{E}[K_\sigma]$ is made explicit. Furthermore, according to Theorem 51, the mean gain of the closed loop is given by

$$\|T(G_\sigma, K_\sigma)\|_{MP} = \sum_{k=0}^{\infty} \max\left\{\left|\bar{t}_1^{\mathbb{E}[K_\sigma]}(k)\right|, \left|\bar{t}_2^{\mathbb{E}[K_\sigma]}(k)\right|\right\}. \tag{3.21}$$

Therefore, the closed-loop is stable, it maps bounded mean inputs to bounded mean outputs, if T_1 and T_2 are stable systems, i.e. their impulse responses are absolute summable. From (3.20), it is clear that T_1 and T_2 are stable if and only if $\mathbb{E}[K_\sigma]$ stabilizes $\mathbb{E}[G_\sigma]$. Therefore, (3.19) reduces to

$$\inf_{K_\sigma \text{ stabilizing}} \|T(G_\sigma, K_\sigma)\|_{MP} = \inf_{\substack{\bar{K} \text{ stabilizing } \mathbb{E}[G_\sigma] \\ \bar{K} \text{ LTI mean representation}}} \sum_{k=0}^{\infty} \max\left\{\left|\bar{t}_1^{\bar{K}}(k)\right|, \left|\bar{t}_2^{\bar{K}}(k)\right|\right\}. \tag{3.22}$$

It is worth noting that the inf in (3.22) is taken over $\bar{K}$s that stabilize $\mathbb{E}[G_\sigma]$ and they are mean representation of some MLSS. Invoking the Youla-Kucera parameterization, $\bar{K}$ stabilizes $\mathbb{E}[G_\sigma]$ if and only if

$$\bar{K} = (Y - MQ)(X - NQ)^{-1}, \tag{3.23}$$

for some stable Q, where

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I$$

is the doubly coprime factorization of the LTI mean representation $\mathbb{E}[G_\sigma]$. This parameterization is known to make the controller synthesis, and in particular (3.22), convex in the Youla parameter. More precisely, using (3.23),

$$T\left(\mathbb{E}[G_\sigma], \bar{K}\right) = H + UQV,$$

where $H := \begin{bmatrix} H_1 & H_2 \end{bmatrix}$, U, and $V := \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ are stable systems depending on $\mathbb{E}[G_\sigma]$. Obviously, the impulse responses T_1 and T_2 are convex in Q. We further need to make sure that $\bar{K}$ is a mean representation of some MLSS. To this end, note that the mean representation of K_σ, $\mathbb{E}[K_\sigma]$, is given by

$$\begin{aligned} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}(k+1) &= \bar{A}_C \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}(k) + \bar{B}_C \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}(k), \\ \begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix}(k) &= \bar{C}_C \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}(k) + \bar{D}_C \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}(k), \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} \bar{A}_C &= \begin{bmatrix} p_{11}A_1^C & p_{12}A_2^C \\ p_{21}A_1^C & p_{22}A_2^C \end{bmatrix}, \\ \bar{B}_C &= \begin{bmatrix} p_{11}B_1^C & p_{12}B_2^C \\ p_{21}B_1^C & p_{22}B_2^C \end{bmatrix}, \\ \bar{C}_C &= \begin{bmatrix} C_1^C & 0 \\ 0 & C_2^C \end{bmatrix}, \bar{D}_C = \begin{bmatrix} D_1^C & 0 \\ 0 & D_2^C \end{bmatrix}, \\ \xi_i(k) &= \mathbb{E}\left[x_C(k)\mathbf{1}_{\sigma(k)=i}\right]. \end{aligned}$$

From this, it is obvious that the D-matrix of the mean representation $\mathbb{E}[K_\sigma]$, $\bar{D}_C$, is diagonal. This proves to be also sufficient for an LTI system to be a mean representation when the switching sequence is Independently Identically Distributed (IID) as stated in the following theorem:

Theorem 52 *Consider a MLSS (3.18) with σ IID. That is $\Pr(\sigma(k+1) = i | \sigma(k) = j) = p_i$, for all $k \in \mathbb{Z}_+$. Then the D-matrix of its mean representation (3.24), mapping two inputs to two outputs, is diagonal. Conversely, any LTI system mapping two inputs to two outputs with diagonal D-matrix is a mean representation of some MLSS in the form of (3.18).*

Proof. Here we prove the converse part. We will show that if σ is IID any LTI system with diagonal D-matrix mapping two inputs to two outputs can be written as the mean representation of some MLSS. First, notice that if σ is IID then $p_{11} = p_{12} = p_1$ and $p_{21} = p_{22} = p_2$. Now, let $X = \begin{bmatrix} p_1^{-1}I & 0 \\ -p_2 I & p_1 I \end{bmatrix}$. Then, apply the the coordinate transformation $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = X \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ on (3.24). The state-space matrices of the transformed system is given by

$$\begin{aligned}
X\bar{A}_C X^{-1} &= \begin{bmatrix} p_1 A_1^C + p_2 A_2^C & p_1^{-1} A_2^C \\ 0 & 0 \end{bmatrix}, \\
X\bar{B}_C &= \begin{bmatrix} B_1^C & B_2^C \\ 0 & 0 \end{bmatrix}, \\
\bar{C}_C X^{-1} &= \begin{bmatrix} p_1 C_1^C & 0 \\ p_2 C_2^C & p_1^{-1} C_1^C \end{bmatrix}, \\
\bar{D}_C &= \begin{bmatrix} D_1^C & 0 \\ 0 & D_2^C \end{bmatrix}.
\end{aligned}$$

From this, it is easy to see that $\zeta_2 = 0$ and hence one can reduce the order of the system and find an equivalent state-space representation as

$$\mathbb{E}\left[K_\sigma\right] = \left[\begin{array}{c|cc} p_1 A_1^C + p_2 A_2^C & B_1^C & B_2^C \\ \hline p_1 C_1^C & D_1^C & 0 \\ p_2 C_2^C & 0 & D_2^C \end{array}\right]. \tag{3.25}$$

Now, given any LTI system, $\bar{K}$, mapping two inputs to two outputs with diagonal D-matrix as

$$\bar{K} = \left[\begin{array}{c|cc} A^K & B_1^K & B_2^K \\ \hline C_1^K & D_1^K & 0 \\ C_2^K & 0 & D_2^K \end{array}\right],$$

one can choose the state-space matrices of K_σ as

$$\begin{aligned}
A_i^C &= A^K, B_i^C = B_i^K, \\
C_i^C &= \frac{1}{p_i} C_i^K, D_i^C = D_i^K.
\end{aligned}$$

Then, the mean representation of K_σ, given in (3.25) matches $\bar{K}$. ■

In the light of above theorem, if σ is IID, the inf in (3.22) should be taken over the stabilizing $\bar{K}$ with diagonal D-matrix. Notice that this condition is, in general, hard to enforce. But, since the D-matrix of $\mathbb{E}\left[G_\sigma\right]$ is diagonal as well, this condition, as shown in [59] when considering a different problem, is satisfied if and only if the D-matrix of the Youla parameter in (3.23) is diagonal.

Theorem 53 *For the MLSS (3.1) with IID σ*

$$\inf_{K_\sigma \ stabilizing} \|T(G_\sigma, K_\sigma)\|_{MP} = \inf_{\substack{Q \ stable \\ \bar{q}(0) \ diagonal}} \sum_{k=0}^{\infty} \max \left\{ \left| \bar{t}_1^Q(k) \right|, \left| \bar{t}_2^Q(k) \right| \right\}, \tag{3.26}$$

where $\{q(k)\}_{k=0}^{\infty}$ is the impulse response of Q and $\left\{\bar{t}_i^Q(k)\right\}_{k=0}^{\infty}$ is the impulse response of $T_i = H_i + UQV_i$, for $i = 1, 2$.

We point out that (3.26) can be computed with arbitrary accuracy. Indeed, it can be cast as a linear program and hence is tractable. Similar type of constrained problems have been dealt with in the past in the context of optimal disturbance rejection for periodic and multirate systems in [14] and [60]. If σ is a Markov process but not IID, then $\bar{q}(0)$ being diagonal is only a necessary condition and not sufficient in general. Hence, (3.26) results in a lower bound of the achievable performance. The sufficient conditions needed to be enforced on $\bar{K}$ such that its is a mean representation of some MLSS is the subject of our future research.

3.4 Summary

In this chapter, we introduced the notion of the stochastic l_∞ gain for LSS. This gain captures the peak to peak performance of the system when the switching sequence is a random process with a given distribution. We provided an exact expression for computing this gain. We further studied the mean performance of MLSS. The mean performance is characterized in terms of the LTI mean representation of the plant. Furthermore, we considered the problem of mean performance optimal control synthesis. In the case when the switching sequence is IID, this problem is reduced to a convex optimization. This optimization can be solved with arbitrary accuracy using linear programming and hence is tractable.

Part II

Systems with Cone Constraints

Chapter 4

Systems with Positive Inputs

4.1 Introduction

The study of systems with positivity constraints is well justified as they appear in many fields when modelling nonnegative entities such as mass, density, volume, etc. as illustrated in the following example:

Example 54 *The tumor-immune interaction can be modeled as*

$$\begin{aligned}\dot{x} &= -\mu_C x \ln\left(\frac{x}{x_\infty}\right) - \gamma xy - \kappa xu,\\ \dot{y} &= \mu_I \left(x - \beta x^2\right) y - \delta y + \alpha,\end{aligned}$$

where y stands for the immunocompetent cells, x is tumor volume, u is the chemotherapy agent, and μ_C, x_∞, γ, κ, μ_I, δ, and α are constant parameters [34]. The phase portrait of this system is shown in Figure 4.1 for $u = 0$. This system has three equilibria away from the origin marked by asterisks in the figure. The state variables x and y remain nonnegative for nonnegative initial condition and inputs.

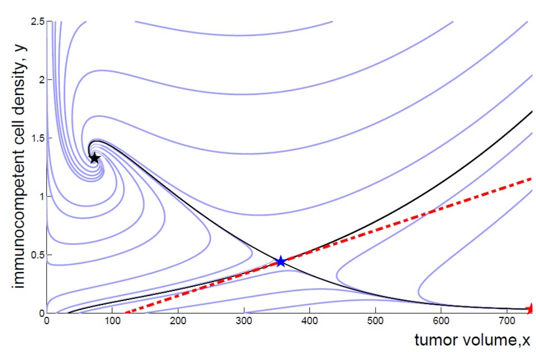


Figure 4.1: Phase portrait of tumor immune interaction

In this part, we are interested in characterizing and optimizing the l_∞ gain of linear systems that contain positivity type of constraints. Two cases are considered: when the input to the system is positive and when the system itself is positive. The former is studied in this chapter while the latter is considered in the next chapter. As an example for the first case, consider the positive nonlinear system in Example 54. If one linearizes this tumor-immune interaction model around one of its equilibria, the linearized model is no longer a positive system as it is not linearized about the origin. However, its input, the chemotherapy agent, remains positive and hence fit into this class of systems.

In this chapter, we assume that the input is restricted to be in the positive cone of l_∞, denoted by l_∞^+, and seek to characterize the induced norm from l_∞^+ to l_∞. That is, for a given (not necessarily positive) linear system G, we are interested to find $\sup_t \|(Gu)(t)\|_\infty$, where $0 \leq u(k) \leq 1$ (the inequalities are taken component wise) for all nonnegative integers k. We obtain an exact characterization of this norm (the induced norm from l_∞^+ to l_∞) in terms of the standard l_∞ induced norms of appropriately defined subsystems which is particularly easy to calculate in the case of LTI systems. We emphasize that no positivity assumption is made on the system itself. We further consider the more general asymmetric input signals and characterize the input output gain of such systems. More precisely, for two real numbers a and b, we compute $\sup_t \|(Gu)(t)\|_\infty$, where $a \leq u(k) \leq b$ for all nonnegative integers k. As an application of the above developments, we consider a filtering problem in which the signal to be estimated, s, is known to live in a positive cone, i.e. $s \in l_\infty^+$. In general, just designing a filter to minimize the standard l_∞ induced norm of the operator from signal to the estimation error will be conservative. Instead, we can use the apriori knowledge of positiveness of the signal by considering the same problem with l_∞^+ to l_∞ induced norm.

Based on this development, we consider the model matching problem to show that time-varying linear or nonlinear control or filtering does not improve the performance with respect to this norm for LTI systems. Also, synthesizing an LTI controller to optimize the l_∞^+ to l_∞ induced norm reduces to linear programming. We further generalize the results to the case of mixed input signals when there are inputs both in l_∞^+ and l_∞. As an example, we consider the aforementioned filtering problem and solve it when the signal is positive and bounded and there also exists noise which is only bounded but not necessarily positive.

4.2 Background and Notation

For any $M = [m_{ij}] \in \mathbb{R}^{n\times m}$, $\|M\|_1 = \max_i \sum_{j=1}^m |m_{ij}|$, $\|M\|_\infty = \max_j \sum_{i=1}^n |m_{ij}|$, and its null space is denoted by $Null(M)$. Also, associated to M, we define two matrices $M^+ = \left[m_{ij}^+\right] \in \mathbb{R}^{n\times m}$ and $M^- = \left[m_{ij}^-\right] \in \mathbb{R}^{n\times m}$ as

$$m_{ij}^+ = 0 \vee m_{ij}, m_{ij}^- = 0 \vee -m_{ij},$$

where $\vee$ stands for the max operator. That is, for two real numbers a and b, $a \vee b := \max\{a, b\}$. We refer to M^+ and M^- as the positive decomposition M and it can be easily verified that $M = M^+ - M^-$. Given a sequence $y = \{y(k)\}_{k=1}^\infty$ where $y(k) \in \mathbb{R}^n$, for $k \in \mathbb{Z}_+$, one can define its positive decomposition into two non-negative sequences y^+ and y^- in an analogous way. In this chapter, we are interested in the positive cone of l_∞^n which is denoted by l_∞^{n+}. This set is defined by

$$l_\infty^{n+} = \left\{ \{y(k)\}_{k=1}^{\infty} \in l_\infty^n : y_i(k) \geq 0, k \in \mathbb{Z}_+, i = 1, ..., n \right\},$$

where $y_i(k)$ is the i^{th} entry of vector $y(k) \in \mathbb{R}^n$. In other words, l_∞^{n+} is the set of bounded non-negative sequences. By $\mathcal{B}(l_\infty^{n+}, \varepsilon)$ ($\mathcal{B}(l_\infty^{n}, \varepsilon)$), for $\varepsilon > 0$, we mean the ball of radius ε in l_∞^{n+} (l_∞^n).

Let $\mathcal{L}_{TV}^{n\times m}$ be the space of all linear, causal, and bounded operators, $T : l_\infty^m \to l_\infty^n$. That is, for any $x, y \in l_\infty^m$, $T(x+y) = Tx + Ty$, $P_k T P_k u = T P_k u$, for $\forall k \in \mathbb{Z}_+$, and

$$\|T\| := \sup_{u \neq 0} \frac{\|Tu\|_\infty}{\|u\|_\infty} < +\infty, \tag{4.1}$$

where P_k is the truncation operator defined by

$$P_k x = (x_0, x_1, ..., x_{k-1}, 0, 0, ...).$$

Also, denote by $\mathcal{L}_{TI}^{n\times m}$ the subspace of all $T \in \mathcal{L}_{TV}^{n\times m}$ such that $\Lambda T = T\Lambda$,where Λ is the delay operator

$$\Lambda x = \Lambda(x_0, x_1, ...) = (0, x_0, x_1, ...), \text{ for } \forall x \in l_\infty^m.$$

It is well-known that any $T \in \mathcal{L}_{TV}^{n\times m}$ can be represented by a lower triangular infinite dimensional matrix

$$T = [T(i,j)]_{i\geq j} = \begin{bmatrix} T(0,0) & 0 & 0 & \cdots \\ T(1,0) & T(1,1) & 0 & \cdots \\ T(2,0) & T(2,1) & T(2,2) & \\ \vdots & & & \ddots \end{bmatrix}, \tag{4.2}$$

where $T(i,j) \in \mathbb{R}^{n\times m}$ for all $i, j \in \mathbb{Z}_+$, $i \geq j$. Moreover, (4.1) defines a norm on $\mathcal{L}_{TV}^{n\times m}$ and

$$\|T\| = \sup_{i\in\mathbb{Z}_+} \left\| \begin{bmatrix} T(i,0) & T(i,1) & \cdots & T(i,i) \end{bmatrix} \right\|_1. \tag{4.3}$$

Also, one can think of the positive decomposition of T into $T^+ = [T^+(i,j)]_{i\geq j} \in \mathcal{L}_{TV}^{n\times m}$ and $T^- = [T^-(i,j)]_{i\geq j} \in \mathcal{L}_{TV}^{n\times m}$.

In [7], the authors introduced the normed space $\mathcal{L}_0^{m\times n}$ whose elements, $G \in \mathcal{L}_0^{m\times n}$, can be represented by upper

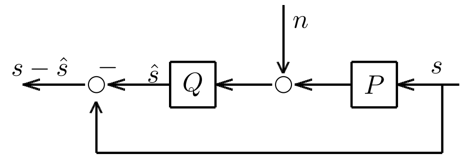


Figure 4.2: Filtering problem

triangular infinite dimensional matrices

$$G = \begin{bmatrix} G(0,0) & G(0,1) & G(0,2) & \cdots \\ 0 & G(1,1) & G(1,2) & \cdots \\ 0 & 0 & G(2,2) & \\ \vdots & & & \ddots \end{bmatrix},$$

where $G(i,j) \in \mathbb{R}^{m\times n}$ for all $i, j \in \mathbb{Z}_+$ and $j \geq i$. Moreover, $\mathcal{L}_0^{m\times n}$ is equipped with a norm, $\|.\|_{\mathcal{L}_0}$,

$$\|G\|_{\mathcal{L}_0} = \sum_i \|\mathcal{C}[G]_i\|_\infty,$$

where $\mathcal{C}[G]_i$ is the i^{th} column of G. It was shown that $\mathcal{L}_0^{m\times n}$ is the pre-dual of $\mathcal{L}_{TV}^{n\times m}$ with pairing $\langle T, G\rangle := Trace(TG)$. Furthermore, $\|T\| = \sup_{\|G\|_{\mathcal{L}_0}\leq 1} \langle T, G\rangle$.

4.3 The Plus Norm Computation

In this chapter, we are interested in linear systems whose input is positive. More precisely, for $T \in \mathcal{L}_{TV}^{n\times m}$, define the functional (norm) $\|.\|_+ : \mathcal{L}_{TV}^{n\times m} \to \mathbb{R}$ as

$$\|T\|_+ = \sup_{\substack{u\neq 0 \\ u\in l_\infty^{m+}}} \frac{\|Tu\|_\infty}{\|u\|_\infty}. \tag{4.4}$$

Intuitively speaking, this functional (induced norm), similarly to l_1 norm for LTI systems, gives the peak to peak ratio of the output to input when the input is restricted to a positive cone. Note that l_∞^+ is not a linear space, however (4.4) is indeed a norm and thus is referred to as the *plus norm* henceforth. It is obvious that the plus norm is dominated by the l_∞ induced norm. As an example, consider the filtering problem depicted in Figure 4.2 where the input to a (stable) plant P is to be estimated. Suppose s belongs to the ball of l_∞^+ and there is no noise for now, i.e. $n = 0$. It is of interest to design the filter Q to minimize the worst case estimation error, $s - \hat{s}$. Therefore, one needs to minimize the worst-case input-output gain of the map $I - QP$ which is the map from input s to the estimation error $s - \hat{s}$. Clearly, just designing a filter to minimize the standard l_∞ induced norm of this operator is in general conservative. Instead, we can use the apriori knowledge of positiveness of the input signal by considering the same problem with l_∞^+ to l_∞ induced norm. In what follows, one of our goals is to characterize this newly defined norm (4.4) and find tractable expressions to compute it.

We develop expressions to calculate the plus norm in terms of the standard l_∞ induced norm of the system. For the simplicity of presentation, we mainly focus on Multi-Input Single-Output (MISO) systems. By doing so, we will not lose any generality for our purposes since any $T \in \mathcal{L}_{TV}^{n\times m}$ can be written as

$$T = \begin{bmatrix} T_1 \\ \vdots \\ T_n \end{bmatrix}, \tag{4.5}$$

where $T_i \in \mathcal{L}_{TV}^{1\times m}$ for $i \in \{1, 2, ..., n\}$ and it is straight forward to show that $\|T\| = \max_i \|T_i\|$, and $\|T\|_+ = \max_i \|T_i\|_+$. In fact by definition,

$$\|T\|_+ = \sup_{\substack{u\neq 0 \\ u\in l_\infty^{m+}}} \frac{\|Tu\|_\infty}{\|u\|_\infty} = \max_i \sup_{\substack{u\neq 0 \\ u\in l_\infty^{m+}}} \frac{\|T_i u\|_\infty}{\|u\|_\infty} = \max_{i\in\{1,\dots,n\}} \|T_i\|_+ . \tag{4.6}$$

Therefore, we mainly state and prove our results for MISO system and note that the extension to MIMO case follows from (4.6). The next lemma connects the plus norm to the standard l_∞ norm of its positive decomposition.

Lemma 55 *Consider a MISO LTV system T with n inputs, $T \in \mathcal{L}_{TV}^{1\times m}$. Then*

$$\|T\|_+ = \max\left\{\left\|T^+\right\|, \left\|T^-\right\|\right\}. \tag{4.7}$$

Proof. By the definition of the plus norm we have $\|T\|_+ = \sup_{\substack{k\in\mathbb{Z}_+ \\ u\in\mathcal{B}\left(l_\infty^+,1\right)}} |y(k)|$, where

$$|y(k)| = \left|\sum_{j=0}^{k} T(k,j)\, u(j)\right| = \left|\sum_{j=0}^{k}\sum_{r=0}^{m} t_r(k,j)\, u_r(j)\right|,$$

where $t_r(k,j)$ is the r^{th} entry of row vector $T(k,j) = [t_1(k,j), t_2(k,j), ..., t_m(k,j)]$. Given $k \in \mathbb{Z}_+$, to maximize $|y(k)|$, u should be chosen in a way to make $y(k)$ either as large (positive) as possible or as small (negative) as possible. In other words, for $k \in \mathbb{Z}_+$,

$$\max_u |y(k)| = \max\left\{\left|\max_u y(k)\right|, \left|\min_u y(k)\right|\right\}. \tag{4.8}$$

First, consider the case of maximizing $y(k)$, $\max_u y(k)$. To make $y(k)$ as positive as possible, it is obvious that one needs to set $u_r(j) = 1$ if $t_r(k,j) \geq 0$ and $u_r(j) = 0$ if $t_r(k,j) < 0$. That is, $\max_u y(k) = \sum_{j=0}^{k}\sum_{r=0}^{m}(t_r(k,j) \vee 0) = \sum_{j=0}^{k}\|T^+(k,j)\|$. Next, to minimize $y(k)$, one needs to set $u_r(j) = 1$ if $t_r(k,j) < 0$ and $u_r(j) = 0$ if $t_r(k,j) \geq 0$. This implies, $\min_u y(k) = -\sum_{j=0}^{k}\sum_{r=0}^{m}(-t_r(k,j) \vee 0) = -\sum_{j=0}^{k}\|T^-(k,j)\|$. Hence, by (4.8) we have

$$\max_u |y(k)| = \max\left\{\sum_{j=0}^{k}\left\|T^+(k,j)\right\|, \sum_{j=0}^{k}\left\|T^-(k,j)\right\|\right\}.$$

Taking the sup with respect to $k \in \mathbb{Z}_+$ in turn implies

$$\|T\|_+ = \sup_{\substack{k\in\mathbb{Z}_+ \\ u\in\mathcal{B}\left(l_\infty^+,1\right)}} |y(k)| = \max\left\{\left\|T^+\right\|, \left\|T^-\right\|\right\},$$

where we have used the fact that T^+ and T^- are MISO positive operators and $\sup_{k\in\mathbb{Z}_+}\sum_{j=0}^{k}\|T^+(k,j)\| = \|T^+\|$ and $\sup_{k\in\mathbb{Z}_+}\sum_{j=0}^{k}\|T^-(k,j)\| = \|T^-\|$. ■

This lemma provides an exact expression for computation of $\|T\|_+$. Another expression for $\|T\|_+$ which fits our purposes in later sections is presented next.

Theorem 56 *Let $T \in \mathcal{L}_{TV}^{1\times m}$. Then,*

$$\|T\|_+ = \sup_k \frac{1}{2}\left(\sum_{j=0}^{k}\sum_{r=1}^{m}|t_r(k,j)| + \left|\sum_{j=0}^{k}\sum_{r=1}^{m}t_r(k,j)\right|\right), \tag{4.9}$$

where $t_r(k,j)$ is the r^{th} entry of row vector $T(k,j) = [t_1(k,j), t_2(k,j), ..., t_m(k,j)]$.

Proof. First, we will show that for given $k \in \mathbb{Z}_+$

$$\frac{1}{2}\left(\sum_{j=0}^{k}\sum_{r=1}^{m}|t_r(k,j)| + \left|\sum_{j=0}^{k}\sum_{r=1}^{m}t_r(k,j)\right|\right) = \max\left\{\sum_{j=0}^{k}\left\|T^+(k,j)\right\|, \sum_{j=0}^{k}\left\|T^-(k,j)\right\|\right\}. \tag{4.10}$$

Without loss of generality assume $\sum_{j=0}^{k}\|T^+(k,j)\| \geq \sum_{j=0}^{k}\|T^-(k,j)\|$. The other case, can be handled similarly. This assumptions implies

$$\sum_{j=0}^{k}\sum_{r=0}^{m}t_r(k,j) \geq 0, \tag{4.11}$$

and that the right hand side of (4.10) $\sum_{j=0}^{k}\|T^+(k,j)\|$. Furthermore, by (4.11), the left hand side of (4.10) can be simplified as:

$$\begin{aligned}
\frac{1}{2}\left(\sum_{j=0}^{k}\sum_{r=1}^{m}|t_r(k,j)| + \left|\sum_{j=0}^{k}\sum_{r=1}^{m}t_r(k,j)\right|\right) &= \frac{1}{2}\left(\sum_{j=0}^{k}\sum_{r=1}^{m}|t_r(k,j)| + \sum_{j=0}^{k}\sum_{r=1}^{m}t_r(k,j)\right) \\
&= \frac{1}{2}\sum_{j=0}^{k}\sum_{r=1}^{m}\left[|t_r(k,j)| + t_r(k,j)\right] \\
&= \sum_{j=0}^{k}\sum_{r=1}^{m}(t_r(k,j)\vee 0) = \sum_{j=0}^{k}\left\|T^+(k,j)\right\|.
\end{aligned}$$

Hence, (4.10) holds. Now, by Lemma 77, taking sup with respect to k from both sides of (4.10) completes the proof.

■

In dealing with LTI systems, (4.9) can be simplified and linked to the usual l_1 (l_∞ induced) norm of the system. Before presenting the results for LTI case, we need to recall that the λ-transform for $T \in \mathcal{L}_{TI}^{n\times m}$ with impulse response

$\{T(k)\}_{k=0}^{\infty}$ is defined by $\hat{T}(\lambda) = \sum_{k=0}^{\infty} \lambda^k T(k)$, for λ's such that the series converges. The following holds true:

Corollary 57 *For a MISO LTI system $T \in \mathcal{L}_{TI}^{1\times m}$,*

$$\|T\|_+ = \frac{1}{2}\left[\|T\| + \left|\hat{T}(1)\,\mathbf{1}\right|\right], \tag{4.12}$$

where $\mathbf{1}$ *is the vector of ones.*

Proof. The proof follows similarly to the proof of Theorem 56 and hence is omitted here. For the SISO case, one can also refer to [61, Proof of Theorem 5]. ■

4.4 Model Matching Problems

In this section, we consider a generic model matching problem

$$\inf_{Q} \|H - UQV\|_+, \tag{4.13}$$

where H, U, and V are stable LTI systems and show that this problem with the norm $\|.\|_+$ is indeed convex and tractable. Moreover, we will show that time varying compensation, $Q \in \mathcal{L}_{TV}$, can not outperform time invariant compensation, $Q \in \mathcal{L}_{TI}$. That is,

$$\inf_{Q\in\mathcal{L}_{TI}} \|H - UQV\|_+ = \inf_{Q\in\mathcal{L}_{TV}} \|H - UQV\|_+.$$

Let $H = \begin{bmatrix} H_1 \\ \vdots \\ H_m \end{bmatrix}$ and $U = \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix}$, where $U_i, H_i \in \mathcal{L}_{TI}^{1\times n}$ for some integers m and n. The following corollary is a direct consequence of Corollary 57:

Corollary 58 *For the model matching problem (4.13), we have*

$$\inf_{Q\in\mathcal{L}_{TI}} \|H - UQV\|_+ = \inf_{Q} \max_{i\in\{1,2,\ldots,m\}} \frac{1}{2}\left[\|H_i - U_iQV\| + \left|\hat{H}_i(1)\,\mathbf{1} - \hat{U}_i(1)\,\hat{Q}(1)\,\hat{V}(1)\,\mathbf{1}\right|\right] \tag{4.14}$$

Note that (4.14) is a linear programming (LP) problem and the optimal value can be found with arbitrary accuracy using methods in [62].

Example 59 *Consider the model matching problem (4.13) with the following:*

$$H = \left[\begin{array}{c|c} \begin{pmatrix} 0.15 & -0.3 \\ 0.07 & 0.4 \end{pmatrix} & \begin{pmatrix} 0.08 & -0.42 \\ 0 & -0.3 \end{pmatrix} \\ \hline \begin{pmatrix} 0.01 & 0.9 \end{pmatrix} & \begin{pmatrix} 0.8 & -0.7 \end{pmatrix} \end{array}\right],$$

$$U = \left[\begin{array}{c|c} \begin{pmatrix} 0.2 & 0.07 \\ -0.5 & 0.2 \end{pmatrix} & \begin{pmatrix} -0.12 \\ -0.22 \end{pmatrix} \\ \hline \begin{pmatrix} 0.65 & 0.8 \end{pmatrix} & -0.8 \end{array}\right],$$

$$V = \left[\begin{array}{c|c} \begin{pmatrix} -0.4 & -0.06 \\ 0.02 & 0.3 \end{pmatrix} & \begin{pmatrix} 0.3 & 0.13 \\ -0.18 & 0.5 \end{pmatrix} \\ \hline \begin{pmatrix} -0.4 & 0.3 \end{pmatrix} & \begin{pmatrix} 0.5 & 0.4 \end{pmatrix} \end{array}\right].$$

For this problem, we have

$$\inf_Q \|H - UQV\| \simeq 1.646,$$

and

$$\inf_Q \|H - UQV\|_+ \simeq 0.946.$$

Notice that the optimal values for the standard l_1 greater than that of the plus norm. Also, it is worth mentioning that the minimizer of the standard l_1 problem does not necessarily minimize the plus norm.

As indicated above, the general, multi-block, model matching problem can be solved via the abstract LP methods in [62]. These primal-dual methods lead to solutions which perform arbitrarily close to the optimal cost, within any prescribed degree of accuracy. However, for single block problems, one can say more about the problem. Indeed, as we elaborate below, we use the standard duality approach of [63] or [62] to obtain exact solutions which also reveal the FIR structure of the optimal solutions. This feature of the norm $\|.\|_+$ is similar to that of the standard l_1 problem.

4.4.1 On Exact Solutions

Herein, we consider the one block problem [63] and, to avoid a lengthy exposition, we treat only the SISO case. The results hold true for MIMO as well. In the previous part, we linked the plus norm to the l_1 norm and the DC gain of the system. Here, invoking duality theory, we will derive some important properties of the optimal solution for the model matching problem. A key in applying the duality approach of [63] and [62] is characterizing the primal and dual spaces. To this end, for a sequence $x = \{x(k)\}_{k=0}^{\infty}$, define two sequences $x^+ = \{x^+(k)\}_{k=0}^{\infty}$ and

$x^{-} = \{x^{-}(k)\}_{k=0}^{\infty}$ by

$$\begin{aligned} x^{+}(k) &= x(k) \vee 0, \\ x^{-}(k) &= -x(k) \vee 0. \end{aligned}$$

Clearly, $x = x^{+} - x^{-}$ and we refer to such a decomposition as the positive decomposition. Also, (with some abuse of notation,) define the plus norm of the sequence x as

$$\|x\|_{+} = \max\left\{\sum_{k=0}^{\infty} x^{+}(k), \sum_{k=0}^{\infty} x^{-}(k)\right\}, \tag{4.15}$$

whenever the summations are finite. It is straight forward to show that the space of sequences with finite plus norm is a normed linear space and we denote it by $\tilde{l}_1$. The following lemma characterizes the dual space of $\tilde{l}_1$:

Lemma 60 *The dual space of $\tilde{l}_1$ is denoted by $\tilde{l}_\infty$ and is the space of all bounded sequences y with the norm*

$$\|y\|_{\tilde{l}_\infty} = \sup_{\|x\|_{+} \le 1} \left|\sum_{k=0}^{\infty} y(k)\, x(k)\right| = \left\|y^{+}\right\|_{\infty} + \left\|y^{-}\right\|_{\infty},$$

where $y = y^{+} - y^{-}$ is the positive decomposition of y.

Proof. It can be easily verified that any given $y \in \tilde{l}_\infty$ defines a bounded functional on the space of $\tilde{l}_1$ with the pairing $\langle y, x\rangle = \sum_{k=0}^{\infty} y(k)\, x(k)$, for any $x \in \tilde{l}_1$. Conversely, as $\tilde{l}_1$ possesses a Schauder basis, any functional f on $\tilde{l}_1$ gives rise to an element $y \in \tilde{l}_\infty$ with $y(k)$ given as the action of f on the k^{th} basis vector. It remains to show the induced norm of the functional. To this end, let $y \in \tilde{l}_\infty$. Then, $\|y\|_{\tilde{l}_\infty} = \sup_{\|x\|_{+} \le 1} \left|\sum_{k=0}^{\infty} y(k)\, x(k)\right|$. Let $y = y^{+} - y^{-}$ be the positive decomposition of y. Then,

$$\sum_{k=0}^{\infty} y(k)\, x(k) = \sum_{k} \left[y^{+}(k)\, x^{+}(k) + y^{-}(k)\, x^{-}(k)\right] - \sum_{k} \left[y^{-}(k)\, x^{+}(k) + y^{+}(k)\, x^{-}(k)\right].$$

Therefore, it can be easily verified that

$$\sum_{k=0}^{\infty} y(k)\, x(k) \le \max\left\{\begin{array}{c} \|y^{+}\|_{\infty} \sum_{k} x^{+}(k) + \|y^{-}\|_{\infty} \sum_{k} x^{-}(k) \\ \|y^{-}\|_{\infty} \sum_{k} x^{+}(k) + \|y^{+}\|_{\infty} \sum_{k} x^{-}(k) \end{array}\right\}.$$

And since $\|x\|_{+} \le 1$, we have

$$\left|\sum_{k=0}^{\infty} y(k)\, x(k)\right| \le \left\|y^{+}\right\|_{\infty} + \left\|y^{-}\right\|_{\infty}.$$

Now, given $\varepsilon > 0$, let $k_1 \neq k_2$ such that

$$\begin{aligned}\left\|y^+\right\|_\infty - \varepsilon &\leq y^+(k_1) = y(k_1) \leq \left\|y^+\right\|_\infty, \\ \left\|y^-\right\|_\infty - \varepsilon &\leq y^-(k_2) = -y(k_2) \leq \left\|y^-\right\|_\infty.\end{aligned}$$

Now, let $x^{opt} = \{x^{opt}(k)\}_{k=0}^{\infty}$ be a sequence of zeros except at k_1 and k_2 with the values of

$$\begin{aligned}x^{opt}(k_1) &= 1, \\ x^{opt}(k_2) &= -1.\end{aligned}$$

Clearly, $\|x^{opt}\|_+ \leq 1$ and

$$\sum_{k=0}^{\infty} y(k)\, x^{opt}(k) \geq \left\|y^+\right\|_\infty + \left\|y^-\right\|_\infty - 2\varepsilon.$$

■

The problem of interest is

$$\inf_Q \|H - UQ\|_+, \tag{4.16}$$

where H and U are stable SISO LTI systems. Further, we assume that U does not have any zero on the unit circle and, for simplicity, its unstable zeros are of multiplicity one. Let $\{a_i\}_{i=1}^N$ be the set of (unstable) zeros of U in the unit disk, i.e. $\hat{U}(a_i) = 0$. Then, a stable LTI system R can be written as $R = UQ$ if and only if $\hat{R}(a_i) = 0$ for $i = 1, 2, ..., N$. Therefore, (4.16) reduces to

$$\inf_R \|H - R\|_+, \text{ subject to } \hat{R}(a_i) = 0 \text{ for } i = 1, 2, ..., N. \tag{4.17}$$

Also, notice that the space of stable LTI systems equipped with the plus norm is isomorphic to $\tilde{l}_1$ and (4.17) can be viewed as a minimum distance problem in $\tilde{l}_1$. Let $r = \{r(k)\}_{k=0}^{\infty}$ and $h = \{h(k)\}_{k=0}^{\infty}$ be the impulse responses of H and R. Also, define the sequences

$$\begin{aligned}\bar{a}_i &= \left\{1, \mathrm{Re}(a_i), \mathrm{Re}\left(a_i^2\right), ...\right\}, \\ \tilde{a}_i &= \left\{0, \mathrm{Im}(a_i), \mathrm{Im}\left(a_i^2\right), ...\right\}.\end{aligned}$$

Then, (4.17) is equivalent to

$$\inf_{r \in M} \|h - r\|_+, \tag{4.18}$$

where $M = \left\{r \in \tilde{l}_1 : \ \langle \bar{a}_i, r \rangle = \langle \tilde{a}_i, r \rangle = 0, i = 1, ..., N\right\}$. Using the standard duality approach the following can now be shown as in [63]

Theorem 61 *The optimal value of (4.16) is given by*

$$\max_{\{\alpha_i,\beta_i\}_{i=1}^N} \sum_{i=1}^N \alpha_i \operatorname{Re}\left[\hat{H}(a_i)\right] + \beta_i \operatorname{Im}\left[\hat{H}(a_i)\right],$$

subject to

$$\begin{aligned} &\mu_1 \geq 0, \mu_2 \geq 0, \mu_1 + \mu_2 \leq 1 \\ -&\mu_2 \leq \sum_{i=1}^N \alpha_i \bar{a}_i(k) + \beta_i \tilde{a}_i(k) \leq \mu_1, \text{ for } k = 1, 2, ..., J, \end{aligned}$$

where J is a pre-computable index that depends only on a_i's. Moreover, an optimal solution $\Phi_0 = H - UQ_0$ to the original problem always exists for some Q_0 and it is FIR of length J

We note that the above is a finite dimensional LP and that Φ_0 can be easily obtained from its solution using alignment, or by directly solving the primal problem which is, after all, a finite dimensional LP. Also note that the constraints in the maximization in the above theorem come directly from the size constraint

$$\left\| \sum_{i=1}^N \alpha_i \bar{a}_i + \beta_i \tilde{a}_i \right\|_{\tilde{l}_\infty} \leq 1,$$

on the dual functional.

4.4.2 Linear vs. Nonlinear

Herein, we prove that time varying Q does not improve performance, which can then be used to establish that the same holds for smooth nonlinear Q. In particular we have the following.

Theorem 62 *Let $H, U,$ and V be LTI systems. Then,*

$$\inf_{Q \in \mathcal{L}_{TI}} \|H - UQV\|_+ = \inf_{Q \in \mathcal{L}_{TV}} \|H - UQV\|_+ .$$

Proof. This proof is the adaptation of the results of [7] to our problem. We start the proof by showing for any given stable $Q \in \mathcal{L}_{TV}$

$$\left\|H - U\Lambda^{-k} Q \Lambda^k V\right\|_+ \leq \|H - UQV\|_+ .$$

This holds since

$$\begin{aligned} \|H - UQV\|_+ &= \sup_{u \in l_\infty^{n+}, u \neq 0} \frac{\|(H - UQV)\, u\|_\infty}{\|u\|_\infty} \\ &\geq \sup_{u \in l_\infty^{n+}, u \neq 0} \frac{\left\|(H - UQV)\, \Lambda^k u\right\|_\infty}{\left\|\Lambda^k u\right\|_\infty} = \left\|\Lambda^{-k} (H - UQV)\, \Lambda^k\right\|_+ , \end{aligned}$$

which in turn equals $\left\|H - U\Lambda^{-k}Q\Lambda^{k}V\right\|_{+}$ as H, U, and V are LTI and commute with the delay operator. Now, define $Q_N = \frac{1}{N}\sum_{k=0}^{N-1}\Lambda^{-k}Q\Lambda^{k}$. Using triangle inequality, it follows that for any $N \in \mathbb{Z}_+$,

$$\left\|H - UQ_NV\right\|_{+} \leq \left\|H - UQV\right\|_{+}.$$

It is argued in [7], [64], and [55] that $\{Q_N\}_{N=0}^{\infty}$ has a weak* convergent subsequence, denote it by $\{Q_{N_k}\}_{k=0}^{\infty}$. That is, $Q_{N_k} \overset{weak^*}{\to} Q_{LTI}$, where $Q_{LTI} \in \mathcal{L}_{TI}$ is stable. Obviously, for any $X \in \mathcal{L}_0^+$ with $\|X\|_{\mathcal{L}_0} \leq 1$ it holds that

$$\langle H - UQ_{N_k}V, X\rangle \to \langle H - UQ_{LTI}V, X\rangle.$$

It can be easily verified that

$$\left\|H - UQ_{LTI}V\right\|_{+} = \sup_{\substack{X\in\mathcal{L}_0^+ \\ \|X\|_{\mathcal{L}_0}\leq 1}} \langle H - UQ_{LTI}V, X\rangle.$$

Now, for $\varepsilon > 0$, let $X \in \mathcal{L}_0^+$ such that $\|X\|_{\mathcal{L}_0} = 1$ and

$$\left\|H - UQ_{LTI}V\right\|_{+} - \varepsilon \leq \langle H - UQ_{LTI}V, X\rangle \leq \left\|H - UQ_{LTI}V\right\|_{+}.$$

Notice that,

$$\langle H - UQ_{N_k}V, X\rangle \leq \left\|H - UQ_{N_k}V\right\|_{+}\|X\|_{\mathcal{L}_0} = \left\|H - UQ_{N_k}V\right\|_{+}.$$

Hence,

$$\langle H - UQ_{LTI}V, X\rangle = \lim_{k\to\infty}\langle H - UQ_{N_k}V, X\rangle \leq \lim\inf_{k\to\infty}\left\|H - UQ_{N_k}V\right\|_{+},$$

and consequently,

$$\left\|H - UQ_{LTI}V\right\|_{+} - \varepsilon \leq \lim\inf_{k\to\infty}\left\|H - UQ_{N_k}V\right\|_{+}.$$

Since, this inequality holds for any ε, we have

$$\left\|H - UQ_{LTI}V\right\|_{+} \leq \lim\inf_{k\to\infty}\left\|H - UQ_{N_k}V\right\|_{+} \leq \left\|H - UQV\right\|_{+},$$

and this completes the proof. ■

Similarly as in [8], one can show that nonlinear smooth Q cannot outperform linear Q.

4.5 Mixed Signals

In the previous section, we focused on the l_∞ gain of the output when the input is restricted to the positive cone l_∞^+. In this section, we consider a more general case when only part of the input is positive, i.e. $u \in l_\infty^{n_1+} \times l_\infty^{n_2}$. To motivate this problem, we give the following example related to filtering:

Consider the problem depicted in Figure 4.2, where $s \in \mathcal{B}\left(l_\infty^{1+}, 1\right)$ is the input to the (stable) plant P and $n \in \mathcal{B}\left(l_\infty^{1}, b\right)$, for some $b \geq 0$, is the measurement noise. The interest is to design a filter Q such that the difference between the input signal, s, and its estimate $\hat{s}$ is minimized in the l_∞ sense. That is, the problem amounts to

$$\inf_Q \sup_{\substack{s\in\mathcal{B}\left(l_\infty^{1+},1\right)\\ n\in\mathcal{B}\left(l_\infty^{1},1\right)}} \left\| \begin{bmatrix} I - QP & -bQ \end{bmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \right\|_\infty .$$

Generally, given $H_1 \in \mathcal{L}_{TI}^{1\times m_1}$ and $H_2 \in \mathcal{L}_{TI}^{1\times m_2}$, if $u = \left(u_1^T, u_2^T\right)^T \in l_\infty^{m_1+} \times l_\infty^{m_2}$, from the definition of the norm it follows that

$$\sup_{u\in l_\infty^{m_1+}\times l_\infty^{m_2}} \frac{\left\| \begin{bmatrix} H_1 & H_2 \end{bmatrix} u \right\|_\infty}{\|u\|_\infty} = \|H_1\|_+ + \|H_2\| .$$

Specializing this to the abovementioned filtering problem, we have

$$\inf_Q \sup_{\substack{s\in\mathcal{B}\left(l_\infty^{1+},1\right)\\ n\in\mathcal{B}\left(l_\infty^{1},b\right)}} \|s-\hat{s}\|_\infty = \inf_Q \left[b\,\|Q\| + \|I-QP\|_+\right].$$

It should be noted that, as before, it can be similarly argued that nonlinear smooth Q's offer no advantage over LTI Q's. However, if non-smooth Q's are allowed, there is a possibility of improving performance, e.g. see [65] and [66] using the invariant set methods. In particular, it is of interest to know if thresholding results in a better performance. More precisely, any LTI solution Q obtained by our methods can be used to generate a simple non-smooth (thresholding) estimator $Q_{NL} = \Upsilon Q$ where

$$(\Upsilon x)(k) = \begin{cases} x(k), \text{ if } x(k) \geq 0 \\ 0, \text{ if } x(k) < 0 \end{cases}, \text{ for } x \in l_\infty.$$

Clearly, such a Q_{NL} does not perform worse than Q as it keeps the estimate of Q if it is non-negative and sets it to zero if negative. However, as stated in the following proposition, it does not perform strictly better either.

Proposition 63 *Let Υ be the thresholding operator. Then*

$$\inf_{Q\in\mathcal{L}_{TI}} \left[b\,\|Q\| + \|I-QP\|_+\right] = \inf_{Q \text{ nonlinear smooth}} \left[b\,\|\Upsilon Q\| + \|I-\Upsilon QP\|_+\right].$$

Proof. Note that Υ can be approximated arbitrarily closely by a smooth function

$$\left(\Upsilon^{\delta}_{\text{smooth}} x\right)(k) = \begin{cases} x(k) & \text{if} \quad x(k) \geq \delta \\ \frac{1}{4\delta}\left(x(k)+\delta\right)^2 & \text{if} \quad -\delta \leq x(k) < \delta \\ 0 & \text{if} \quad x(k) < -\delta \end{cases},$$

where $\delta > 0$. It is easy to verify that $\Upsilon^{\delta}_{\text{smooth}}$ is smooth and

$$\left\|\Upsilon^{\delta}_{\text{smooth}} - \Upsilon\right\|_{+} = \left\|\Upsilon^{\delta}_{\text{smooth}} - \Upsilon\right\| = \delta.$$

Therefore, given $\varepsilon > 0$ and a stable nonlinear smooth Q, there exists $\delta > 0$ such that

$$b\left\|\Upsilon^{\delta}_{\text{smooth}} Q\right\| + \left\|I - \Upsilon^{\delta}_{\text{smooth}} QP\right\|_{+} \leq b\left\|\Upsilon Q\right\| + \left\|I - \Upsilon QP\right\|_{+} + \varepsilon. \tag{4.19}$$

Now, note that as $\Upsilon^{\delta}_{\text{smooth}} Q$ is smooth it admits a linearization $\bar{Q}$ such that

$$\sup_{0<\|f\|_{\infty}\leq\alpha} \frac{\left\|\left(\Upsilon^{\delta}_{\text{smooth}} QP - \bar{Q}P\right) f\right\|_{\infty}}{\|f\|_{\infty}} < \varepsilon,$$

and

$$\sup_{0<\|f\|_{\infty}\leq\alpha} \frac{\left\|\left(\Upsilon^{\delta}_{\text{smooth}} Q - \bar{Q}\right) f\right\|_{\infty}}{\|f\|_{\infty}} < \varepsilon$$

for some $\alpha > 0$. Therefore,

$$\begin{aligned} & b\left\|\Upsilon^{\delta}_{\text{smooth}} Q\right\| + \left\|I - \Upsilon^{\delta}_{\text{smooth}} QP\right\|_{+} \\ \geq\ & b \sup_{0<\|f\|_{\infty}\leq\alpha} \frac{\left\|\left(\Upsilon^{\delta}_{\text{smooth}} Q\right) f\right\|_{\infty}}{\|f\|_{\infty}} + \sup_{\substack{0<\|f\|_{\infty}\leq\alpha \\ f\in l^{+}_{\infty}}} \frac{\left\|\left(I - \Upsilon^{\delta}_{\text{smooth}} QP\right) f\right\|_{\infty}}{\|f\|_{\infty}} \\ \geq\ & b\left\|\bar{Q}\right\| + \left\|I - \bar{Q}P\right\|_{+} - (1+b)\,\varepsilon. \end{aligned} \tag{4.20}$$

Therefore, from (4.19) and (4.20) we have

$$\inf_{\bar{Q}\in\mathcal{L}_{TV}} b\left\|\bar{Q}\right\| + \left\|I - \bar{Q}P\right\|_{+} \leq \inf_{Q \text{ smooth nonlinear}} b\left\|\Upsilon Q\right\| + \left\|I - \Upsilon QP\right\|_{+}.$$

Further, similarly to Theorem 62, one can argue that the LTV Q's cannot lead a better performance than LTI Q's. And hence, we have

$$\inf_{Q\in\mathcal{L}_{TI}} \left[b\left\|Q\right\| + \left\|I - QP\right\|_{+}\right] = \inf_{Q \text{ smooth nonlinear}} \left[b\left\|\Upsilon Q\right\| + \left\|I - \Upsilon QP\right\|_{+}\right].$$

■

Note that the above proposition asserts that even a nonlinear smooth Q followed by a thresholding Υ does not perform better than LTI.

4.6 Asymmetric Signals

In this subsection, we present results for a more general case when the input signal is asymmetric and its lower and upper bounds are time-varying. To this end, let $a, b \in l_\infty^m$ be two bounded sequences and suppose that the input satisfies

$$a \leq u \leq b,$$

where the inequalities are taken component wise. Then the following can be easily proved:

Proposition 64 *For a given $T \in \mathcal{L}_{TV}^{1\times m}$ with positive decomposition $T = T^+ - T^-$,*

$$\sup_{a\leq u\leq b} \|Tu\|_\infty = \max\left\{\left\|T^+b - T^-a\right\|_\infty, \left\|T^+a - T^-b\right\|_\infty\right\}.$$

Notice that, the above expression requires the positive decomposition of the operator. Similarly, to the proof of Theorem 56, one can show the following:

Theorem 65 *For given $T \in \mathcal{L}_{TV}^{1\times m}$ and $a, b \in l_\infty^m$,*

$$\begin{aligned} &\sup_{a\leq u\leq b} \|Tu\|_\infty \\ &= \frac{1}{2}\sup_k \left\{\left|\sum_{j=0}^{k}\sum_{r=1}^{m} t_r(k,j)\left(a_r(j) + b_r(j)\right)\right| + \sum_{j=0}^{k}\sum_{r=1}^{m} \left|t_r(k,j)\right|\left(b_r(j) - a_r(j)\right)\right\}, \end{aligned} \tag{4.21}$$

where $t_r(k,j)$ is the r^{th} entry of the row vector $T(k,j) = \begin{bmatrix} t_1(k,j) & \cdots & t_m(k,j)\end{bmatrix}$ and $a_r(j)$ ($b_r(j)$) is the r^{th} component of $a(j) \in R^m$ ($b(j) \in R^m$).

To relate (4.21) to the standard l_∞ norm of the operator, for given $x = \{x(j)\}_{j=0}^\infty \in l_\infty^m$, define the bounded operator Π_x as

$$\Pi_x = \begin{bmatrix} diag(x(0)) & & & \\ & diag(x(1)) & & \\ & & diag(x(2)) & \\ & & & \ddots \end{bmatrix}.$$

Then, we note that, the first on the right hand side of (4.21) is the sum of the i^{th} row of the matrix representation of the operator $T\left(\Pi_a+\Pi_b\right)$. Also, the second term is the l_1 norm of the k^{th} row of the operator $T\left(\Pi_a-\Pi_b\right)$. Therefore,

$$\sup_{a\leq u\leq b}\|Tu\|_\infty=\frac{1}{2}\sup_k\left\{\left|\mathcal{R}\left[T\left(\Pi_a+\Pi_b\right)\right]_k\mathbf{1}\right|+\left\|\mathcal{R}\left[T\left(\Pi_b-\Pi_a\right)\right]_k\right\|\right\},$$

where $\mathbf{1}$ is the vector of ones with appropriate dimension and $\mathcal{R}\left[T\left(\Pi_a+\Pi_b\right)\right]_k$ $\left(\mathcal{R}\left[T\left(\Pi_b-\Pi_a\right)\right]_k\right)$ is the k^{th} row of the infinite dimensional matrix representation of the operator $T\left(\Pi_a+\Pi_b\right)$ $\left(T\left(\Pi_b-\Pi_a\right)\right)$. For LTI systems this expression can be further simplified.

Corollary 66 *Let $a=\{\alpha,\alpha,...\}$ and $b=\{\beta,\beta,...\}$ be constant sequences in l_∞^m, with $\alpha,\beta\in\mathbb{R}^m$. Then, for $T\in\mathcal{L}_{TI}^{1\times m}$,*

$$\sup_{a\leq u\leq b}\|Tu\|_\infty=\frac{1}{2}\left[\left|\hat{T}\left(1\right)\left(\alpha+\beta\right)\right|+\left\|T\left(\Pi_a-\Pi_b\right)\right\|\right].$$

Given the above results, LP can be used to compute system's performance and solve for optimal model matching, similarly to the previous sections.

4.7 Summary

In this chapter, we considered linear systems whose inputs are restricted to be in the positive cone of l_∞. This led to introducing the plus norm, which is the induced norm from l_∞^+ to l_∞. We presented an exact characterization of this norm for both LTV and LTI systems. Further, for the LTI systems, we gave an expression for the plus norm in terms of the standards l_1 norm of the system and its DC gain. As an application, a filtering problem was studied. Furthermore, based on this development, we considered the model matching problem and showed that time-varying linear or nonlinear control or filtering does not improve the performance with respect to the plus norm, and synthesizing an optimal controller for minimizing the plus norm is a LP.

Chapter 5

Positive Systems

5.1 Introduction

In this chapter, we address the case where the positivity constraints are imposed on the systems. From the input-output perspective, an externally positive system is the one whose output is in the positive l_∞ cone when the input is in this cone, starting from zero initial condition. As we point out, if such a constraint is imposed on the closed loop map, finding an optimal controller is a linear programming problem and hence tractable [62]. Also, if the model matching problem for LTI systems is considered, time varying linear or nonlinear compensation cannot outperform LTI even if external positivity is enforced. Furthermore, if internal positivity is sought, we show that a dynamic controller offers no advantage over a static one as far as l_1, l_∞, or $\mathcal{H}_\infty$ performance is concerned. Therefore, the abovementioned results can be readily used to obtain an optimal (static) state feedback controller or output feedback for special cases. We note that, designing an optimal output feedback controller (which is static) is a harder problem and in general leads to a bilinear program. In certain cases, however, when the measurement matrix satisfies certain conditions, such problem is shown to reduce to a linear program as will be discussed.

5.2 External Positivity

An operator $T \in \mathcal{L}_{TV}^{n\times m}$ is said to be externally positive if for all $i, j \in \mathbb{Z}_+$, $i \geq j$, $T(i,j) \in \bar{\mathbb{R}}_+^{n\times m}$, where $\bar{\mathbb{R}}_+^{n\times m}$ is the closure of $\mathbb{R}_+^{n\times m}$ in standard topology. The set of such operators is denoted by $\mathcal{L}_{TV}^{n\times m+}$. In analogous way, we also define $\mathcal{L}_{TI}^{n\times m+}$ and $\mathcal{L}_0^{m\times n+}$. Our first result is that designing a stabilizing controller such that the closed loop system is externally positive can be cast as a convex optimization. Consider a general control problem where $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} : \begin{pmatrix} w \\ u \end{pmatrix} \rightarrow \begin{pmatrix} z \\ y \end{pmatrix}$ is the generalized plant; w and u are the exogenous and control input; z and y are the regulated and measured output, respectively. The problem of interest is to find a controller $K : y \rightarrow u$ that stabilizes the plant, minimizes the effects of w on z, and makes the map from w to z externally positive. Such a problem can be converted to the following LP:

$$\mu := \inf_{Q \text{ stable}} \|H - UQV\|,$$

for some stable H, U, and V [63], [62], subject to

$$H - UQV \geq 0, \tag{5.1}$$

where the inequality in (5.1) is taken component-wise on the impulse response of $H - UQV$ or its lower triangular representation. Although it is an infinite dimensional optimization, its solution can be obtained with arbitrary accuracy, through finite dimensional LP. For problems of this sort, we refer to [62] and [58]. Moreover, as is discussed in Appendix 7.1, nonlinear smooth Q's do not outperform LTI ones. In what follows, we present an example of the filtering problem with positivity constraints both on signals and systems.

Example 67 *Consider the abovementioned filtering problem depicted in Figure 4.2 where $s \in \mathcal{B}\left(l_\infty^{1+}, 1\right)$ and $n \in \mathcal{B}\left(l_\infty^{1}, b\right)$. The objective is to design a filter Q that minimizes the estimation error and produces a positive estimate in the absence of noise. That is, if $n = 0$ and $s \in l_\infty^{+}$ then $\hat{s} \in l_\infty^{+}$. Based on our developments in the previous section, one can argue that this problem amounts to*

$$\inf_{Q} \sup_{\substack{s \in \mathcal{B}\left(l_\infty^{1+}, 1\right) \\ n \in \mathcal{B}\left(l_\infty^{1}, b\right)}} \left\| \begin{bmatrix} I - QP & Q \end{bmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \right\|_\infty = \inf_{Q} \left\{ \|I - QP\|_+ + \|bQ\| \right\}, \tag{5.2}$$

subject to

$$QP \geq 0. \tag{5.3}$$

For this example, let $b = 0.3$ and

$$P = \left[\begin{array}{c|c} \begin{pmatrix} -0.07 & 0.15 \\ -0.78 & 0.12 \end{pmatrix} & \begin{pmatrix} -0.25 \\ -0.26 \end{pmatrix} \\ \hline \begin{pmatrix} -0.5 & -0.1 \end{pmatrix} & (0.5) \end{array} \right].$$

Then

$$\inf_{Q} \left\{ \|I - QP\|_+ + \|bQ\| \right\} \simeq 0.715.$$

It is worth noting that if instead of the plus norm, one uses the standard l_1 norm, a different performance is achieved. Indeed,

$$\inf_{Q} \left\{ \|I - QP\| + \|bQ\| \right\} \simeq 0.850.$$

For this particular filtering example, it can be shown (see Appendix 7.2) that solving (5.2) without the constraint (5.3) does not lead a better performance, contrary to what one may expect. Therefore, if it is of interest to have a positive estimate $\hat{s}$ even in the presence of the noise, after solving (5.2) for Q without the constraint (5.3), one can replace Q with ΥQ without changing the performance. This is also discussed in Appendix 7.2.

5.3 Internal Positivity

One can also think in terms of a state-space realization of T,

$$T:\begin{cases} x\left(t+1\right)=Ax\left(t\right)+Bu\left(t\right) \\ y\left(t\right)=Cx\left(t\right)+Du\left(t\right) \end{cases}, \tag{5.4}$$

where x, u, and y are state, input and output, respectively; and A, B, C, and D are matrices of appropriate dimensions.

Definition 68 *An operator T with state-space realization of the form (5.4) is internally positive if and only if the output and the states are nonnegative whenever the input and the initial condition are nonnegative.*

It can be shown that the above definition is equivalent to matrices A, B, C, and D having nonnegative entries [37]. Obviously, internal positivity implies external positivity but the converse is not true, in general. In state-space, there is a simple way to calculate the l_1 norm (l_∞ induced norm) of an externally positive LTI system, G, with state-space matrices (A,B,C,D). As reported in [45], one has

$$\|G\|=\left\|C\left(I-A\right)^{-1}B\mathbf{1}+D\mathbf{1}\right\|_\infty,$$

where $\mathbf{1}$ is a column vector of compatible dimension with all entries equal to one. Moreover, the following holds:

Lemma 69 *(discrete-time counterpart of Lemma 2 of [45]) If G is internally positive then $\|G\|<\gamma$ for some $\gamma>0$ if and only if there exists $\nu\in\mathbb{R}_+^n$ such that*

$$A\nu+B\mathbf{1}_{n_w}<\nu,\qquad C\nu+D\mathbf{1}_{n_w}<\gamma\mathbf{1}_{n_z},$$

where n_w, n_z, and n are the number of inputs, outputs, and states, respectively.

Let

$$G=\left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right), K=\left(\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right), \tag{5.5}$$

then, the map T from w to z is given by

$$T\left(G,K\right)=\left(\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right), \tag{5.6}$$

where

$$\begin{aligned}
A_{cl} &= \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix}, B_{cl} = \begin{bmatrix} B_1 + B_2 D_k D_{21} \\ B_k D_{21} \end{bmatrix}, \\
C_{cl} &= \left[C_1 + D_{12} D_k C_2, D_{12} C_k \right], D_{cl} = D_{11} + D_{12} D_k D_{21}.
\end{aligned}$$

Now, we present a new result regarding the optimal control synthesis for such systems. The next theorem addresses a problem which was previously reported as an open problem in [67].

Theorem 70 *For $\gamma > 0$, if there exists a controller (5.5) of order n_k such that the closed loop system (5.6) is internally positive, stable, and has l_1 norm less than γ ($\|T(G,K)\| < \gamma$), then there exists a static controller $\bar{K}$ such that $T\left(G,\bar{K}\right)$ is also positive, internally stable, and $\left\|T\left(G,\bar{K}\right)\right\| < \gamma$.*

Proof. Suppose a controller K with state-space matrices as in (5.5) yields to a positive closed loop system $T(G,K)$ with $\|T(G,K)\| < \gamma$. The result follows by direct calculations showing $T\left(G,\bar{K}\right)$ has the desired properties where

$$\bar{K} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_k \end{array} \right).$$

Indeed, since $\|T(G,K)\| < \gamma$, by Lemma 69, there should exists $\nu_1 \in \mathbb{R}_+^n$, $\nu_2 \in \mathbb{R}_+^{n_k}$ such that

$$\begin{aligned}
A_{cl} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} + B_{cl} \mathbf{1}_{n_w} &< \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \\
C_{cl} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} + D_{cl} \mathbf{1}_{n_w} &< \gamma \mathbf{1}_{n_z}.
\end{aligned}$$

Since the closed loop (more precisely $B_2 C_k$ and $D_{12} C_k$) and ν_2 are non-negative, from the above inequalities, it holds that

$$\begin{aligned}
\left(A + B_2 D_k C_2\right) \nu_1 + \left(B_1 + B_2 D_k D_{21}\right) \mathbf{1}_{n_w} &< \nu_1, \\
\left(C_1 + D_{12} D_k C_2\right) \nu_1 + \left(D_{11} + D_{12} D_k D_{21}\right) \mathbf{1}_{n_w} &< \gamma \mathbf{1}_{n_z}.
\end{aligned}$$

By Lemma 69, the above two inequalities imply $\left\|T\left(G,\bar{K}\right)\right\| < \gamma$. ■

Finding a static controller $K \in \mathbb{R}^{n_u \times n_y}$ where n_u and n_y are the number of control inputs and measured outputs such that $\|T(G,K)\| < \gamma$ is in general a bilinear program stated in the next Proposition. For simplicity, define

$$\hat{A} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \in \mathbb{R}^{(n+n_y)\times(n+n_w)},$$

$$\hat{B} = \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \in \mathbb{R}^{(n+n_y)\times n_u},\ \hat{C} = \begin{bmatrix} C_2 & D_{21} \end{bmatrix} \in \mathbb{R}^{n_y\times(n+n_w)},$$

and assume the following:

Assumption 71 *Suppose C_2 is full row rank.*

Let the set of vectors $\{f_i\}_{i=1}^{n-n_y}$, for $f_i \in \mathbb{R}^n$, span the null space of C_2. Further, let the sets of vectors $\{g_i\}_{i=1}^{n_d}$ and $\{h_i\}_{i=1}^{n_b}$ span the null space of D_{21}, $\hat{B}^T$, respectively, where $g_i \in \mathbb{R}^{n_w}$ and $h_i \in \mathbb{R}^{n+n_y}$; n_d and n_b are the dimensions of the null spaces of D_{21} and $\hat{B}^T$.

Proposition 72 *There exists a static output feedback controller such that $\|T(G,K)\| < \gamma$ and $T(G,K)$ is internally positive if and only if there exist $\nu \in \mathbb{R}_+^n$, $\hat{\mu} \in \bar{\mathbb{R}}_+^{(n+n_y)\times n}$, $E \in \bar{\mathbb{R}}_+^{(n+n_y)\times n_w}$, and a set of vectors $\{\zeta_i\}_{i=1}^{n-n_y}$ such that*

$$\begin{bmatrix} \hat{\mu} & E \end{bmatrix} \mathbf{1}_{n+n_w} \leq \begin{pmatrix} \nu \\ \gamma \mathbf{1}_{n_z} \end{pmatrix}, \tag{5.7}$$

$$h_i^T \hat{A} \begin{bmatrix} \Pi & 0 \\ 0 & I \end{bmatrix} = h_i^T \begin{bmatrix} \hat{\mu} & E \end{bmatrix}, \tag{5.8}$$

$$\hat{A} \begin{bmatrix} \Pi & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} \zeta_i \\ g_i \end{pmatrix} = \begin{bmatrix} \hat{\mu} & E \end{bmatrix} \begin{pmatrix} \zeta_i \\ g_i \end{pmatrix}, \tag{5.9}$$

$$\Pi \zeta_i = f_i. \tag{5.10}$$

where $\Pi = diag(\nu_1, ..., \nu_n)$. In this case, the controller K is given by

$$K = \hat{B}^{-L} \left(\begin{bmatrix} \hat{\mu}\Pi^{-1} & E \end{bmatrix} - A \right) \hat{C}^{-R}, \tag{5.11}$$

where $\hat{B}^{-L}$ and $\hat{C}^{-R}$ are left and right inverses of $\hat{B}$ and $\hat{C}$, respectively.

The proof of this proposition depends heavily on the following standard linear algebra result [68]:

Lemma 73 *Let $\hat{A}$, $\hat{B}$, $\hat{C}$, and X be matrices with appropriate dimensions. Then, there exists a matrix K such that*

$$\hat{A} + \hat{B}K\hat{C} = X, \tag{5.12}$$

if and only if

$$\begin{aligned}\hat{A}f &= Xf,\\ h^T\hat{A} &= h^T X,\end{aligned}$$

for and $f \in Null\left(\hat{C}\right)$ *and* $h \in Null\left(\hat{B}^T\right)$. *In this case,* $K = \hat{B}^{-L}\left(X - \hat{A}\right)\hat{C}^{-R}$, *where* $\hat{B}^{-L}$ *and* $\hat{C}^{-R}$ *are left and right inverses of* $\hat{B}$ *and* $\hat{C}$, *respectively.*

Proof of Proposition 72. According to Lemma 69, $\|T(G,K)\| < \gamma$ and $T(G,K)$ is internally positive for some K if and only if there exist $E_1 \in \bar{\mathbb{R}}_+^{(n+n_y)\times n}$, $E_2 \in \bar{\mathbb{R}}_+^{(n+n_y)\times n_w}$, $\nu \in \mathbb{R}_+^n$, and K such that

$$\hat{A} + \hat{B}K\hat{C} = \begin{bmatrix} E_1 & E_2 \end{bmatrix} \geq 0, \tag{5.13}$$

$$\left(\hat{A} + \hat{B}K\hat{C}\right)\begin{pmatrix} \nu \\ \mathbf{1}_{n_w} \end{pmatrix} < \begin{pmatrix} \nu \\ \gamma\mathbf{1}_{n_z} \end{pmatrix}. \tag{5.14}$$

Define $\Pi := diag\left(\nu_1, \nu_2, ..., \nu_n\right)$, $\hat{\mu} = E_1\Pi \geq 0$, and $E := E_2 \geq 0$. Then, (5.13) and (5.14) simplify to

$$\hat{A} + \hat{B}K\hat{C} = \begin{bmatrix} \hat{\mu} & E \end{bmatrix}\begin{bmatrix} \Pi^{-1} & 0 \\ 0 & I \end{bmatrix}, \tag{5.15}$$

$$\begin{bmatrix} \hat{\mu} & E \end{bmatrix}\mathbf{1}_{n+n_w} \leq \begin{pmatrix} \nu \\ \gamma\mathbf{1}_{n_z} \end{pmatrix}. \tag{5.16}$$

Using Lemma 73, (5.15) has a solution for K if and only if conditions (5.8) and

$$\hat{A}\begin{pmatrix} f_i \\ g_i \end{pmatrix} = \begin{bmatrix} \hat{\mu} & E \end{bmatrix}\begin{bmatrix} \Pi^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{pmatrix} f_i \\ g_i \end{pmatrix},$$

which is equivalent to (5.9) after replacing f_i with $\Pi\zeta_i$. Therefore, the proof is complete by noticing that (5.16) is the same as (5.7) and (5.11) is achieved by pre- and post-multiplying (5.13) by $\hat{B}^{-L}$ and $\hat{C}^{-R}$. ∎

We comment here that (5.7)-(5.10) can be reduced to LP in some special cases. For example, in [69], the state feedback problem is posed as a linear program. In this case, $C_2 = I$ which has the trivial null space of $\{0\}$. This simplifies conditions (5.9) and (5.10) to

$$\hat{A}\begin{pmatrix} 0 \\ g_i \end{pmatrix} = \begin{bmatrix} \hat{\mu} & E \end{bmatrix}\begin{pmatrix} 0 \\ g_i \end{pmatrix},$$

which together with (5.7) and (5.8) is a tractable linear program. This cannot however be done for the general output-feedback problem unless the C_2 matrix satisfies certain conditions as stated in the next corollary.

Corollary 74 *Suppose that the null space of C_2 is invariant under multiplication by invertible diagonal matrices. That is, for any diagonal invertible matrix M,*

$$Mf_i \in span\left\{f_1, ..., f_{n-n_y}\right\}.$$

Then, there exists an static output feedback controller such that $\|T(G,K)\| < \gamma$ and $T(G,K)$ is internally positive if and only if there exist $\nu \in \mathbb{R}_+^n$, $\hat{\mu} \in \bar{\mathbb{R}}_+^{(n+n_y)\times n}$, $E \in \bar{\mathbb{R}}_+^{(n+n_y)\times n_w}$ such that

$$\begin{bmatrix} \hat{\mu} & E \end{bmatrix} \mathbf{1}_{n+n_w} \leq \begin{pmatrix} \nu \\ \gamma \mathbf{1}_{n_z} \end{pmatrix}, \tag{5.17}$$

$$h_i^T \hat{A} \begin{bmatrix} \Pi & 0 \\ 0 & I \end{bmatrix} = h_i^T \begin{bmatrix} \hat{\mu} & E \end{bmatrix}, \tag{5.18}$$

$$\hat{A} \begin{bmatrix} \Pi & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} f_i \\ g_i \end{pmatrix} = \begin{bmatrix} \hat{\mu} & E \end{bmatrix} \begin{pmatrix} f_i \\ g_i \end{pmatrix}. \tag{5.19}$$

Proof. We note that since the null space of C_2 is invariant under multiplication by Π^{-1} and Π, ζ_i satisfies (5.10) if and only if $\zeta_i \in Null\left(C_2\right)$. Therefore, (5.9) is simplified to (5.19). ■

We would like to point out that an important class of output feedback program satisfies the condition in the above corollary. The C_2 matrix for the systems in this class has $n - n_y$ zero columns. This happens, for example, if the n_y measurements are the linear combinations of n_y states and the rest $n - n_y$ states do not enter explicitly in the output equation.

Finally, we would like to remark that results similar to Theorem 70 can be shown for some other performance measures. For instance, the next theorem deals with the case when the performance is measured in l_2 induced sense.

Theorem 75 *If there exists a dynamic controller (5.5) such that the closed loop system (5.6) is internally positive, stable, and has l_2 induced norm less than γ ($\|T(G,K)\|_{l_2-ind} < \gamma$) for some positive γ, then there exists a static controller $\bar{K}$ such that $T\left(G,\bar{K}\right)$ is also internally positive, stable, and $\left\|T\left(G,\bar{K}\right)\right\|_{l_2-ind} < \gamma$.*

Before proving this theorem, we need the following lemmas:

Lemma 76 *Let G and H be non-negative matrices with $0 \leq G \leq H$. Then, $\bar{\sigma}(G) \leq \bar{\sigma}(H)$, where $\bar{\sigma}(.)$ denotes the maximum singular value.*

Proof. Notice that $0 \leq G^T \leq H^T$. Therefore, $0 \leq G^T G \leq H^T H$, [48, Lemma 3], and hence $\rho\left(G^T G\right) \leq \rho\left(H^T H\right)$, [70], where $\rho(.)$ is the spectral radius. ■

Lemma 77 *Given an internally positive G with state-space matrices (A, B, C, D), the following three conditions are equivalent:*

(i) $\|G\|_{l_2-ind} < \gamma$, for some $\gamma > 0$.

(ii) there exists a positive matrix Z of compatible dimension such that

$$AZ + B < Z, \tag{5.20}$$

$$\begin{bmatrix} I & CZ + D \\ Z^T C^T + D^T & \gamma^2 I \end{bmatrix} \text{ is positive definite.} \tag{5.21}$$

(iii) there exists a positive matrix Z of compatible dimension such that

$$ZA + C < Z,$$

$$\begin{bmatrix} I & ZB + D \\ Z^T B^T + D^T & \gamma^2 I \end{bmatrix} \text{ is positive definite.}$$

Proof. We only show the equivalency of (i) and (ii). Notice that since G is internally positive, $\|G\|_{l_2-ind} < \gamma$ if and only if $\left\|\hat{G}(1)\right\|_{l_2-ind} = \bar{\sigma}\left(\hat{G}(1)\right) < \gamma$, where $\hat{G}(1)$ is the DC gain of G. That is, $\|G\|_{l_2-ind} < \gamma$ if and only if

$$\bar{\sigma}\left[C(I - A)^{-1} B + D\right] < \gamma. \tag{5.22}$$

First, suppose (5.22) holds. Since A is non-negative and stable, $(I - A)^{-1}$ is non-negative as well. Therefore, for any positive matrix X, $Y := (I - A)^{-1} X \geq 0$. Moreover, one can choose $X > 0$ such that $Y > 0$. Now, since (5.22) is strict inequality, there exists $\varepsilon > 0$ such that

$$\bar{\sigma}\left[C\left((I - A)^{-1} B + \varepsilon Y\right) + D\right] < \gamma.$$

Let $Z := (I - A)^{-1} B + \varepsilon Y$. Then, $(I - A) Z - B = \varepsilon Y > 0$, and $\bar{\sigma}[CZ + D] < \gamma$ which are equivalent to (5.20) and (5.21), respectively.

Conversely, suppose (5.20) and (5.21) hold. Notice that, (5.20) implies A is Schur stable and $(I - A)^{-1} B < Z$. Therefore, $C(I - A)^{-1} B + D < CZ + D$. By Lemma 76, this implies $\bar{\sigma}\left[C(I - A)^{-1} B + D\right] < \bar{\sigma}[CZ + D]$. Furthermore, (5.21), invoking Schur complement type of argument, implies $\bar{\sigma}[CZ + D] < \gamma$ which completes the proof of the converse. ■

Proof of Theorem 75. Now, to prove the theorem, let $K = \left(\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right)$ be the dynamic controller of some order n_k in the statement of the theorem. We will show $\bar{K} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_k \end{array}\right)$ makes the closed-loop system internally positive, stable, and $\left\|T\left(G,\bar{K}\right)\right\|_{l_2-ind}<\gamma$. We will only show that $\left\|T\left(G,\bar{K}\right)\right\|_{l_2-ind}<\gamma$ as the rest of the proof follows similarly to that of Theorem 70.

Since $\|T\left(G,K\right)\|_{l_2-ind}<\gamma$, according to Lemma 77, there exists $Z=\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}\in\mathbb{R}_+^{(n+n_k)\times n_w}$ such that

$$A_{cl}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}+B_{cl}<\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \tag{5.23}$$

and

$$\begin{bmatrix} I & C_{cl}Z+D_{cl} \\ Z^TC_{cl}^T+D_{cl}^T & \gamma^2 I \end{bmatrix} \text{is positive definite,}$$

where the latter is equivalent to

$$\bar{\sigma}\left(C_{cl}Z+D_{cl}\right)<\gamma, \tag{5.24}$$

One can easily show that (5.23) implies

$$\left(A+B_2D_kC_2\right)Z_1+\left(B_1+B_2D_kD_{21}\right)<Z_1. \tag{5.25}$$

Furthermore, using Lemma 76, (5.24) implies

$$\bar{\sigma}\left(\left(C_1+D_{12}D_kC_2\right)Z_1+\left(D_{11}+D_{12}D_kD_{21}\right)\right)<\gamma, \tag{5.26}$$

since

$$\left(C_1+D_{12}D_kC_2\right)Z_1+\left(D_{11}+D_{12}D_kD_{21}\right)\leq C_{cl}Z+D_{cl}.$$

Invoking Lemma 77, (5.25) and (5.26) yield $\left\|T\left(G,\bar{K}\right)\right\|_{l_2-ind}<\gamma$. ■

5.4 Summary

In this chapter, we considered the positive systems (internal and external) in the context of l_∞ optimization. We showed that if external positivity is imposed on the closed loop map, finding an optimal controller is LP and hence tractable. Furthermore, if internal positivity is desired for the closed loop system, a dynamic controller offers no advantage over a static one. We also solved the static output feedback problem for the case that the null space of the output matrix is invariant under multiplication by diagonal matrices.

Chapter 6

Summary and Future Work

This dissertation was split into two parts. In the first part, the theory of l_1 optimal control was extended to LSS. In the second part, the l_∞ performance and control design for system with positivity constraints were considered.

In Part I, we introduced the class of generalized input-output switching systems. We showed how the worst-case gain of these systems can be cast as LP. Furthermore, any stable LSS can be approximated by a generalized input-output switching system with arbitrary accuracy. Then, we addressed the problems of stability, gain computation, and optimal control synthesis for a general LSS. We showed how these problems can be formulated as LPs. Also, we considered the minimal-gain of LSS and showed that an optimal switching is periodic. Moreover, we introduced the notion of the stochastic l_∞ gain which mimics the standard l_∞ induced norm. We characterized the input-output behavior of MLSS in this metric. We further studied the l_∞ mean performance of MLSS and synthesized controllers with respect to this measure of performance.

In Part II, we dealt with characterization and optimization of the l_∞ gain of linear systems that contain positivity type of constraints. First, we considered the case where only the input is restricted to be in the positive cone of l_∞ and characterized the induced norm from l_∞^+ to l_∞, the plus norm. This allowed us to synthesize optimal controllers in the plus norm sense. Then, we considered both internally and externally positive systems. We pointed out that finding an optimal controller while making the closed-loop externally positive is LP and hence a tractable problem. If, on the other hand, the constraint known as internal positivity is sought, we showed that a dynamic controller offers no advantage over a static one. These results can be used to obtain an optimal (static) state feedback controller. However, designing an optimal output feedback controller (which is static) is in general a bilinear program. We showed that this bilinear program can be reduced to LP, if the null space of the measurement matrix is invariant under multiplication by diagonal matrices, such as in the case when part of the states is measured.

Future Work:

One can extend the results presented here in several directions as follows:

Control Synthesis for LSS:

Based on Corollary 24, the gain computation of a general LSS is reduced to a search over two parameters $\delta \in (0, +\infty)$ and $Q_\sigma \in \mathcal{S}_{IO}$. The computation is not convex in both δ and Q_σ, jointly. It is of interest to investigate if there is an alternative way without this shortcoming. Moreover, the stabilizability results can be used in principle to find the doubly coprime factors of a LSS. It may be interesting to see if the doubly coprime factors of a LSS can be linked to those of its LTI modes, at least for the class of input-output switching systems.

Moreover, when synthesizing controllers we assumed that the controller has the knowledge of the plant's switching sequence. It is of interest to investigate what happens if the controller does not have access to the switching sequence. In this case, one can possibly rewrite P_σ as the upper linear fractional transformation of a nominal LTI system $\bar{P}$ and a switching system Δ_σ which, due to its dependency on σ, is not known to the controller. In this case, one can synthesize a robust controller for the LTI $\bar{P}$ which guarantees the desired performance for all Δ_σ.

Best LTI Approximation of LSS:

Given a stable LSS G_σ, it is interesting to know how closely it can be approximated by a LTI system $\bar{G}$. That is,

$$\gamma = \inf_{\bar{G} \text{ LTI}} \sup_{\sigma} \left\| G_\sigma - \bar{G} \right\| .$$

The interest in this problem arises from the situation when the switching sequence is not known to the controller. In this case, the controller could be synthesized to robustly stabilize the LTI system $\bar{G}$ for any Δ, where $\Delta = G_\sigma - \bar{G}$ and $\|\Delta\| \leq \gamma$. We conjecture that for an input-output LSS $G_\sigma = S_\sigma G S_\sigma^*$, one can fully characterize $\bar{G}$ in terms of the impulse response of G. We would like to see what can be said for general G_σ.

Control Synthesis for LTV:

As mentioned before, we addressed the state feedback control synthesis for LSS and LTV systems while the output feedback is left to be investigated more. Here, we provide some general results on the output feedback case. Although these results in general are not computationally appealing, they are interesting from the theory point of view as they give a unified framework to study any l_p performance. Also, they may be extended in future in a direction to cope with their computations.

Given a sequence of $m \times n$ matrices $X = \{X(t) \in \mathbf{R}^{m \times n}\}_{t=0}^{\infty}$, we defined a linear operator $\widehat{X}$ as

$$\widehat{X} = \begin{bmatrix} X(0) & & & \\ & X(1) & & \\ & & X(2) & \\ & & & \ddots \end{bmatrix}.$$

We make an extensive use of these notations to write LTV systems in an operator form. To this end, consider a LTV system P

$$P:\begin{cases} x\left(t+1\right)=A\left(t\right)x\left(t\right)+B\left(t\right)w\left(t\right) \\ \\ y\left(t\right)=C\left(t\right)x\left(t\right)+D\left(t\right)w\left(t\right) \end{cases} ;x\left(0\right)=x_0, \tag{6.1}$$

where $A\left(t\right)$, $B\left(t\right)$, $C\left(t\right)$, and $D\left(t\right)$ are matrices with appropriate dimensions, for each $t\in\mathbf{Z}_{+}$. One can form operators $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, and $\widehat{D}$ as discussed above and rewrite 6.1 as

$$P:\begin{cases} x=\left(I-\Lambda\widehat{A}\right)^{-1}\Lambda\widehat{B}w+\left(I-\Lambda\widehat{A}\right)^{-1}\bar{x_0} \\ y=\widehat{C}x+\widehat{D}w \end{cases}, \tag{6.2}$$

where $x=\{x\left(t\right)\}_{t=0}^{\infty}$, $w=\{w\left(t\right)\}_{t=0}^{\infty}$, $y=\{y\left(t\right)\}_{t=0}^{\infty}$, and $\bar{x}_0=\{x_0,0,0,...\}$. System P can be thought of as a linear map from $\bar{x_0}$ and w to x and y. Consequently, we define its stability as follows:

Definition 78 *Let $x_0\in\mathbf{R^n}$ and $w=\{w\left(t\right)\}_{t=0}^{\infty}\in\mathcal{V}$, where $\mathcal{V}$ is some vector space. System P in (6.2) is said to be $\mathcal{V}$ to $\mathcal{W}=\mathcal{W}_1\times\mathcal{W}_2$ stable if P maps any input sequence $\{x_0,w\}\in\mathbf{R^n}\oplus\mathcal{V}$ to an output sequence $\{\left(x\left(t\right),y\left(t\right)\right)\}_{t=0}^{\infty}$ in the vector space $\mathcal{W}$, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are two vector spaces.*

One can take $\mathcal{V}$ and $\mathcal{W}$ to be the spaces of the bounded magnitude or bounded energy sequences, i.e. l_∞ and l_2, and study the l_∞ or l_2 $\left(\mathcal{H}_\infty\right)$ performance of the system. In this sense, we provide a unifying framework for studying different types of input-output characteristics of a system.

In the sequel, we will appeal to the next lemma which is a standard linear algebra result, see e.g.[68].

Lemma 79 *Let n and m be positive integers with $m\le n$. Given matrices $A\in\mathbf{R}^{n\times n}$, $B\in\mathbf{R}^{n\times m}$, $C\in\mathbf{R}^{m\times n}$, and $Q\in\mathbf{R}^{n\times n}$, with B^T and C being full row rank, the equation*

$$A+BKC=Q$$

has a solution for K if and only if

$$AN\left(C\right)=QN\left(C\right),$$

$$N\left(B^T\right)^TA=N\left(B^T\right)^TQ.$$

In above $N\left(C\right)$ and $N\left(B^T\right)$ are matrices whose columns span the null spaces of C and B^T, respectively. In this case, a solution for K is given by

$$K=B^{-L}\left(Q-A\right)C^{-R},$$

where B^{-L} and C^{-R} are the left and right inverse of B and C.

Given a LTV system P as in 6.2, its stability is equivalent to the boundedness of $\left(I-\Lambda\widehat{A}\right)^{-1}$. Invoking the

Youla-Kucera parameterization, one can show that $\left(I - \Lambda\widehat{A}\right)^{-1}$ is stable if and only if

$$\widehat{A} = Q\left(I + \Lambda Q\right)^{-1}, \tag{6.3}$$

where Q is some stable LTV system. Hence, the following holds:

Proposition 80 *Consider the LTV system P in 6.2. Suppose, $\widehat{B}$, $\widehat{C}$, and $\widehat{D}$ are bounded. Then, the system P is stable if and only if there exists a stable Q such that*

$$\widehat{A}\left(I + \Lambda Q\right) = Q, \tag{6.4}$$

and

$$\left(I + Q\Lambda\right)\widehat{A} = Q. \tag{6.5}$$

Proof. Equations (6.4) and (6.5) follow immediately from multiplying (6.3) by $(I + \Lambda Q)$ from right or $(I + Q\Lambda)$ from left. ■

Conditions (6.4) and (6.5) are convex and checking them becomes particularly easier in the case of LTI system. We use these two conditions for control synthesis. To this end, consider the generalized plant P given in operator form by

$$P : \begin{cases} x = \Lambda\widehat{A}x + \Lambda\widehat{B_1}w + \Lambda\widehat{B_2}u + \bar{x_0} \\ z = \widehat{C_1}x + \widehat{D_{11}}w + \widehat{D_{12}}u \\ y = \widehat{C_2}x + \widehat{D_{21}}w \end{cases}, \tag{6.6}$$

where w is the exogenous input, u is the control input, z is the regulated output, and y is the measured output.

In the context of control synthesis, first, we want to find a controller, K, which maps the measured output, y, to control input, u, and results in a stable closed loop system. To make the idea more concrete, suppose $u = Ky$ for some K. We emphasize that the only restriction we enforce on K is linearity and causality. That is, K can be represented by an infinite dimensional lower triangular matrix

$$K = \begin{bmatrix} k_{00} & 0 & 0 & 0 & \cdots \\ k_{10} & k_{11} & 0 & 0 & \cdots \\ k_{20} & k_{21} & k_{22} & 0 & \cdots \\ & & & \ddots & \end{bmatrix}.$$

Then, the closed loop system $\phi(P,K)$ is given by

$$\phi(P,K):\begin{cases} x=\Lambda A_{cl}x+\Lambda B_{cl}w+\bar{x_0} \\ z=C_{cl}x+D_{cl}w \end{cases}, \tag{6.7}$$

where

$$\begin{aligned} A_{cl} &= \widehat{A}+\widehat{B_2}K\widehat{C_2}, \\ B_{cl} &= \widehat{B_1}+\widehat{B_2}K\widehat{D_{21}}, \\ C_{cl} &= \widehat{C_1}+\widehat{D_{12}}K\widehat{C_2}, \\ D_{cl} &= \widehat{D_{11}}+\widehat{D_{12}}K\widehat{D_{21}}. \end{aligned}$$

The closed-loop system is a mapping from $\bar{x_0}$ and w to x and z. Therefore, it can be partitioned accordingly as

$$\phi(P,K)=\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}:\begin{pmatrix} \bar{x_0} \\ w \end{pmatrix}\rightarrow\begin{pmatrix} x \\ z \end{pmatrix}. \tag{6.8}$$

According to Proposition 80, K results in a stable closed-loop if and only if conditions (6.4) and (6.5) hold for A_{cl}. Before we state our results on the stabilizability, we need to characterize the null spaces of the linear maps $\widehat{B_2}$ and $\widehat{C_2}$. Recall that $\widehat{C_2}$ is a diagonal operator with entries in the set $\{C_2(t)\}_{t=0}^{\infty}$. For any $t\in\mathbf{Z}_+$, let $N(C_2(t))$ be a matrix whose columns span the right null space of $C_2(t)$. Furthermore, we denote by $\widehat{N(C_2)}$ a diagonal operator with entries in the set $\{N(C_2(t))\}_{t=0}^{\infty}$. Similarly, we can define $\widehat{N\left(B_2^T\right)}^T$ consisting of elements in the left null space of $\widehat{B_2}$ or the right null space of $\widehat{B_2^T}$. Henceforth, we make the following assumption:

Assumption 81 *Operators $\widehat{B_2}$ and $\widehat{C_2}$ have left and right inverses, respectively.*

The necessary and sufficient condition for Assumption 81 is the existence of the left and right inverse of $B_2(t)$ and $C_2(t)$, respectively, for all $t\in\mathbf{Z}_+$. We denote these inverses by $\widehat{B_2^{-L}}$ and $\widehat{C_2^{-R}}$.

Theorem 82 *Given the generalized plant P in (6.6), there exists an stabilizing output feedback control K mapping y to u if and only if there exists a stable LTV Q such that*

$$\widehat{N\left(B_2^T\right)}^T\widehat{A}(I+\Lambda Q)=\widehat{N\left(B_2^T\right)}^T Q, \tag{6.9}$$

and

$$(I+Q\Lambda)\widehat{A}\widehat{N(C_2)}=Q\widehat{N(C_2)}. \tag{6.10}$$

In this case, a stabilizing controller is given by

$$K = \widehat{B_2^{-L}} \left(Q \left(I + \Lambda Q \right)^{-1} - \widehat{A} \right) \widehat{C_2^{-R}}. \tag{6.11}$$

Proof. Equations (6.9) and (6.10) are the direct consequence of Lemma 79 in conjuction with Proposition 80.More precisely, K stabilizes the plant if and only if

$$A_{cl} = Q \left(I + \Lambda Q \right)^{-1} = \left(I + Q \Lambda \right)^{-1} Q. \tag{6.12}$$

Using Lemma 79, given Q, (6.12) has a solution for K if and only if

$$\widehat{N \left(B_2^T \right)}^T A_{cl} = \widehat{N \left(B_2^T \right)}^T Q \left(I + \Lambda Q \right)^{-1},$$
$$A_{cl} \widehat{N \left(C_2 \right)} = \left(I + Q \Lambda \right)^{-1} Q \widehat{N \left(C_2 \right)}.$$

After post and premultiplying these equations by $(I + \Lambda Q)$ and $(I + Q\Lambda)$, we obtain (6.9) and (6.10). ■

Upon substituting (6.11) in (6.7) and direct calculation, one can show that the closed-loop is an affine function of Q. In particular we have the following:

Theorem 83 *The set of all closed-loop maps (6.7), for stabilizing K, is given by*

$$\left\{ \phi \left(P, K \right) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} : K \text{ stabilizing } P \right\} =$$
$$\left\{ H + UQV : Q \text{ stable} \right\},$$

where

$$H = \begin{bmatrix} I & \Lambda M_3 \\ M_2 & M_1 + M_2 \Lambda M_3 \end{bmatrix},$$

$$U = \begin{bmatrix} \Lambda \\ M_4 + M_2 \Lambda \end{bmatrix},$$

$$V = \begin{bmatrix} I & M_5 + \Lambda M_3 \end{bmatrix},$$

and M_i's are diagonal operators given by

$$
\begin{aligned}
M_1 &= \widehat{D_{11}} - \widehat{D_{12}}\widehat{B_2^{-L}}\widehat{A}\widehat{C_2^{-R}}\widehat{D_{21}},\\
M_2 &= \widehat{C_1} - \widehat{D_{12}}\widehat{B_2^{-L}}\widehat{A},\\
M_3 &= \widehat{B_1} - \widehat{A}\widehat{C_2^{-R}}\widehat{D_{21}},\\
M_4 &= \widehat{D_{12}}\widehat{B_2^{-L}},\\
M_5 &= \widehat{C_2^{-R}}\widehat{D_{21}}.
\end{aligned}
$$

According to this theorem, to synthesize an optimal controller to minimize the input-output gain, one needs to solve the convex optimization problem

$$\inf_{Q\, stable} \|H + UQV\|,$$

subject to (6.9) and (6.10). Furthermore, if one wants to enforce any positivity constraint on the closed loop, ϕ, it can be readily done through enforcing linear constraints on Q.

We should mention that a major computational burden of this method is due to (6.9) and (6.10). These equations, although convex, are infinite dimensional optimization and in general not easy to satisfy them exactly. However, finding Q to "almost" satisfy them with arbitrary accuracy is LP and tractable. At this point, it is not clear how tightly (6.9) and (6.10) should be satisfied so the rest of the results still hold. We hope a small-gain like argument helps us in analyzing the situation when we substitute (6.9) and (6.10) with

$$\left\| N\widehat{\left(B_2^T\right)}^T \widehat{A}\left(I + \Lambda Q\right) - N\widehat{\left(B_2^T\right)}^T Q \right\| \leq \epsilon, \tag{6.13}$$

and

$$\left\| \left(I + Q\Lambda\right)\widehat{A}\widehat{N\left(C_2\right)} - Q\widehat{N\left(C_2\right)} \right\| \leq \epsilon, \tag{6.14}$$

for small enough $\epsilon > 0$.

Minimal Gain:

On the subject of the minimal-gain of LSS, Theorem 27 states that an optimal switching is periodic. However, its period or finding what an optimal sequence remained unanswered. This problem can be related to sensor scheduling or controlled sensing and might be easier to handle in the stochastic framework. We solved this problem for the stochastic l_∞ gain of the input-output LSS. It is however an open problem for the general case. This also relates to the filtering problem in Figure 6.1. In this problem one needs to minimize the stochastic gain of $I - \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} S_\sigma^* S_\sigma \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$.

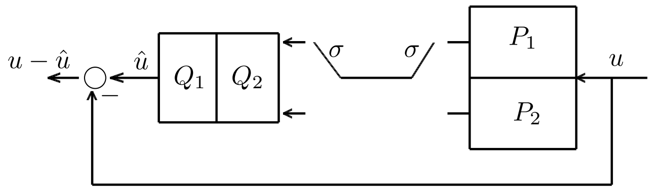


Figure 6.1: Filtering Problem

Systems over Positive Cones:

We defined the plus norm as the induced norm from l_∞^+ to l_∞. One can think of extending our results to other signal spaces for example from l_2^+ to l_2. Also, regarding the internally positive systems, as we showed, static controllers are optimal if the internal positivity with respect to the states of the plant and the controller is enforced. We conjecture that this is also the case when only the internal positivity with respect to the states of the plant is enforced. Moreover, some results similar to those discussed earlier in this section, Control Synthesis for LTV systems, may be useful in synthesizing an optimal controller which enforces the positivity of the states of the plant (and not necessarily the plant).

Chapter 7

Appendix

7.1 Nonlinear vs. Linear in the Presence of Positivity Constraints

In this section, we want to show that for the model matching problem

$$\inf_{Q \text{ stable}} \left\| H - UQV \right\|,$$

subject to

$$H - UQV \geq 0,$$

nonlinear smooth Q's cannot outperform LTI ones. First, we will show that smooth nonlinear Q's cannot outperform LTV Q's. Let Q_{NL} be a smooth nonlinear map. Let $\varepsilon > 0$ be given. Then, there exist a linear map Q_L and $\delta > 0$ such that

$$\sup_{0 < \|f\|_\infty \leq \delta} \frac{\left\| U \left(Q_{NL} - Q_L \right) V f \right\|_\infty}{\|f\|_\infty} < \varepsilon.$$

Now, similarly to the proof of Proposition 63, we have

$$\left\| H - UQ_L V \right\| \leq \left\| H - UQ_{NL} V \right\|.$$

It remains to show that the linearization, Q_L, satisfies the positivity constraints. To this end, let $f \in l_\infty^+$ and $H - UQ_{NL}V \geq 0$ then for given non-negative integer k,

$$\begin{aligned}
&\frac{\delta}{\|f\|_\infty} \left(H - UQ_L V \right) (f)(k) = \left(H - UQ_L V \right) \left(\frac{\delta f}{\|f\|_\infty} \right) (k) \\
&= \left\{ \left(H - UQ_{NL} V \right) + U \left(Q_{NL} - Q_L \right) V \right\} \left(\frac{\delta f}{\|f\|_\infty} \right) (k) \\
&\geq \left[U \left(Q_{NL} - Q_L \right) V \right] \left(\frac{\delta f}{\|f\|_\infty} \right) (k). \qquad (7.1)
\end{aligned}$$

Notice that

$$\left| \left[U \left(Q_{NL} - Q_L \right) V \right] \left(\frac{\delta f}{\|f\|_\infty} \right) (k) \right| \leq \delta \varepsilon.$$

Hence, (7.1) becomes

$$\left(H - UQ_L V\right)(f)(k) \geq -\|f\|_{\infty}\,\varepsilon,$$

and since it holds for any $\varepsilon > 0$, $f \in l_{\infty}^{+}$, and k,

$$H - UQ_L V \geq 0.$$

That is the linearization of a nonlinear map leads a better performance while maintaining the positivity of the closed loop. This linearization may not be time invariant. However, similarly to [8], one can argue LTV compensations cannot do any better than LTI ones and hence in general smooth nonlinear Q's does not lead a better performance that LTI Q's even though the closed loop external positivity is enforced. Finally, as an obvious observation, we note that positivity constraints can be present on any affine linear map of Q for all of the above to hold, i.e., not only to a the same map $H - UQV$. This is the case in Example 67.

7.2 More on the Filtering Problem of Example 67

Define $\nu(Q) := b\|Q\| + \|I - QP\|_{+}$ where b is a positive number. Herein, we will show that

$$\inf_{Q \text{ nonlinear smooth}} \nu(\Upsilon Q) = \inf_{Q \in \mathcal{L}_{TI}} \nu(Q) = \inf_{\substack{Q \in \mathcal{L}_{TI} \\ QP \geq 0}} \nu(Q),$$

where Υ is the thresholding operator. The first equality is proved in Proposition 63. Regarding the second equality, note that we have

$$\inf_{Q \in \mathcal{L}_{TI}} \nu(Q) \leq \inf_{\substack{Q \in \mathcal{L}_{TI} \\ QP \geq 0}} \nu(Q).$$

We will show that

$$\inf_{\substack{Q \in \mathcal{L}_{TI} \\ QP \geq 0}} \nu(Q) \leq \inf_{Q \text{ nonlinear smooth}} \nu(\Upsilon Q). \tag{7.2}$$

To this end, given Q, let $\varepsilon > 0$ and $\Upsilon^{\delta}_{\text{smooth}}$ be the approximation of Υ as defined in the proof of Proposition 63 such that, $\nu\left(\Upsilon^{\delta}_{\text{smooth}} Q\right) \leq \nu(\Upsilon Q) + \varepsilon$. Now, note that $\Upsilon^{\delta}_{\text{smooth}} Q$ is smooth and $\Upsilon^{\delta}_{\text{smooth}} QP \geq 0$. Therefore, by the previous developments (Appendix 7.1), the linearization of $\Upsilon^{\delta}_{\text{smooth}} Q$, denote it by $\bar{Q} \in \mathcal{L}_{TV}$, satisfies $\bar{Q}P \geq 0$ with $\nu\left(\bar{Q}\right) \leq \nu\left(\Upsilon^{\delta}_{\text{smooth}} Q\right)$. Taking inf from the left hand side, we have for any nonlinear smooth Q,

$$\inf_{\substack{\bar{Q} \in \mathcal{L}_{TV} \\ \bar{Q}P \geq 0}} \nu\left(\bar{Q}\right) \leq \nu\left(\Upsilon^{\delta}_{\text{smooth}} Q\right) \leq \nu(\Upsilon Q) + \varepsilon.$$

Since, $\inf_{\substack{Q \in \mathcal{L}_{TI} \\ QP \geq 0}} \nu(Q) = \inf_{\substack{\bar{Q} \in \mathcal{L}_{TV} \\ \bar{Q}P \geq 0}} \nu\left(\bar{Q}\right)$, and ε was arbitrary, (7.2) holds true.

Chapter 8

References

[1] M. Vidyasagar, "Optimal rejection of persistent bounded disturbances," *Automatic Control, IEEE Transactions on*, vol. 31, no. 6, pp. 527–534, 1986.

[2] M. Dahleh and J. Pearson, "l_1-optimal-feedback controllers for discrete-time systems," in *American Control Conference, 1986*. IEEE, 1986, pp. 1964–1968.

[3] ——, "l_1-optimal feedback controllers for mimo discrete-time systems," *Automatic Control, IEEE Transactions on*, vol. 32, no. 4, pp. 314–322, 1987.

[4] J. Doyle, "Structured uncertainty in control system design," in *1985 24th IEEE Conference on Decision and Control*, no. 24, 1985, pp. 260–265.

[5] M. Dahleh and Y. Ohta, "A necessary and sufficient condition for robust bibo stability," *Systems & control letters*, vol. 11, no. 4, pp. 271–275, 1988.

[6] M. Vidyasagar, *Control system synthesis: a factorization approach*. Morgan & Claypool Publishers, 2011.

[7] J. Shamma and M. Dahleh, "Time-varying versus time-invariant compensation for rejection of persistent bounded disturbances and robust stabilization," *Automatic Control, IEEE Transactions on*, vol. 36, no. 7, pp. 838–847, 1991.

[8] M. Dahleh and J. Shamma, "Rejection of persistent bounded disturbances: Nonlinear controllers," *Systems & control letters*, vol. 18, no. 4, pp. 245–252, 1992.

[9] I. Diaz-Bobillo and M. Dahleh, "State feedback l_1-optimal controllers can be dynamic," *Systems & control letters*, vol. 19, no. 2, pp. 87–93, 1992.

[10] J. Shamma, "Nonlinear state feedback for l_1 optimal control," *Systems & control letters*, vol. 21, no. 4, pp. 265–270, 1993.

[11] ——, "Optimization of the l_∞-induced norm under full state feedback," *Automatic Control, IEEE Transactions on*, vol. 41, no. 4, pp. 533–544, 1996.

[12] P. Voulgaris, "On optimal l_∞ to l_∞ filtering," *Automatica*, vol. 31, no. 3, pp. 489–495, 1995.

[13] P. Voulgaris, M. Dahleh, L. Valavani *et al.*, "On slowly varying systems: l_∞ to l_∞ performance and implications to robust adaptive control," in *American Control Conference, 1992*. IEEE, 1992, pp. 1197–1201.

[14] M. Dahleh, P. Voulgaris, L. Valavani *et al.*, "Optimal and robust controllers for periodic and multirate systems," *Automatic Control, IEEE Transactions on*, vol. 37, no. 1, pp. 90–99, 1992.

[15] D. Liberzon, *Switching in systems and control*. Springer, 2003.

[16] H. Lin and P. Antsaklis, "Stability and stabilizability of switched linear systems: a survey of recent results," *Automatic Control, IEEE Transactions on*, vol. 54, no. 2, pp. 308–322, 2009.

[17] Z. Sun and S. S. Ge, *Switched linear systems: Control and design*. Springer, 2005.

[18] M. Fiacchini and M. Jungers, "Necessary and sufficient condition for stabilizability of discrete-time linear switched systems: a set-theory approach," *Automatica*, vol. 50, no. 1, pp. 75–83, 2014.

[19] J. C. Geromel and P. Colaneri, "Stability and stabilization of discrete time switched systems," *International Journal of Control*, vol. 79, no. 07, pp. 719–728, 2006.

[20] J.-W. Lee and G. E. Dullerud, "Uniform stabilization of discrete-time switched and markovian jump linear systems," *Automatica*, vol. 42, no. 2, pp. 205–218, 2006.

[21] F. Blanchini, S. Miani, and F. Mesquine, "A separation principle for linear switching systems and parametrization of all stabilizing controllers," *Automatic Control, IEEE Transactions on*, vol. 54, no. 2, pp. 279–292, 2009.

[22] A. Bhaya, F. Das, and C. Mota, "Equivalence of stability concepts for discrete time-varying systems," *International Journal of Robust and Nonlinear Control*, vol. 4, no. 6, pp. 725–740, 1994.

[23] V. D. Blondel and J. N. Tsitsiklis, "The boundedness of all products of a pair of matrices is undecidable," *Systems & Control Letters*, vol. 41, no. 2, pp. 135–140, 2000.

[24] P. Seiler and R. Sengupta, "An H_∞ approach to networked control," *Automatic Control, IEEE Transactions on*, vol. 50, no. 3, pp. 356–364, 2005.

[25] J. Lee and G. Dullerud, "Optimal disturbance attenuation for discrete-time switched and markovian jump linear systems," *SIAM Journal on Control and Optimization*, vol. 45, p. 1329, 2006.

[26] J. P. Hespanha, "Root-mean-square gains of switched linear systems," *Automatic Control, IEEE Transactions on*, vol. 48, no. 11, pp. 2040–2045, 2003.

[27] M. Margaliot and J. P. Hespanha, "Root-mean-square gains of switched linear systems: A variational approach," *Automatica*, vol. 44, no. 9, pp. 2398–2402, 2008.

[28] J. C. Geromel, P. Colaneri, and P. Bolzern, "Dynamic output feedback control of switched linear systems," *Automatic Control, IEEE Transactions on*, vol. 53, no. 3, pp. 720–733, 2008.

[29] D. Roberson and D. Stilwell, "L_2 gain performance analysis of linear switched systems: fast switching behavior," in *American Control Conference, 2007. ACC'07.* IEEE, 2007, pp. 2084–2089.

[30] O. Costa, M. Fragoso, and R. Marques, *Discrete-time Markov jump linear systems.* Springer Science & Business Media, 2006.

[31] E. Boukas and P. Shi, "H_∞ control for discrete-time linear systems with markovian jumping parameters," in *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, vol. 4. IEEE, 1997, pp. 4134–4139.

[32] H. Chizeck, A. Willsky, and D. Castanon, "Discrete-time markovian-jump linear quadratic optimal control," *International Journal of Control*, vol. 43, no. 1, pp. 213–231, 1986.

[33] O. Costa and M. Fragoso, "Stability results for discrete-time linear systems with markovian jumping parameters," *Journal of mathematical analysis and applications*, vol. 179, no. 1, pp. 154–178, 1993.

[34] U. Ledzewicz, M. Naghnaeian, and H. Schättler, "Optimal response to chemotherapy for a mathematical model of tumor–immune dynamics," *Journal of mathematical biology*, vol. 64, no. 3, pp. 557–577, 2012.

[35] ——, "Dynamics of tumor-immune interaction under treatment as an optimal control problem," *DYNAMICAL SYSTEMS*, pp. 971–980, 2011.

[36] A. Berman and R. J. Plemmons, "Nonnegative matrices," *The Mathematical Sciences, Classics in Applied Mathematics,*, vol. 9, 1979.

[37] L. Farina and S. Rinaldi, *Positive linear systems: theory and applications.* John Wiley & Sons, 2011, vol. 50.

[38] E. Kaszkurewicz and A. Bhaya, *Matrix diagonal stability in systems and computation.* Springer Science & Business Media, 2012.

[39] T. Kaczorek, *Positive 1D and 2D systems.* Springer Science & Business Media, 2012.

[40] W. Haddad, V. Chellaboina, and Q. Hui, *Nonnegative and compartmental dynamical systems.* Princeton University Press, 2010.

[41] E. Fornasini and M. E. Valcher, "Linear copositive lyapunov functions for continuous-time positive switched systems," *Automatic Control, IEEE Transactions on*, vol. 55, no. 8, pp. 1933–1937, 2010.

[42] M. E. Valcher, "Controllability and reachability criteria for discrete time positive systems," *International Journal of Control*, vol. 65, no. 3, pp. 511–536, 1996.

[43] L. Farina, "On the existence of a positive realization," *Systems & Control Letters*, vol. 28, no. 4, pp. 219–226, 1996.

[44] J. Van Den Hof, "Realization of continuous-time positive linear systems," *Systems & control letters*, vol. 31, no. 4, pp. 243–253, 1997.

[45] C. Briat, "Robust stability and stabilization of uncertain linear positive systems via integral linear constraints: $\mathcal{L}_1$-gain and $\mathcal{L}_\infty$-gain characterization," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 17, pp. 1932–1954, 2013.

[46] Y. Ebihara, D. Peaucelle, and D. Arzelier, "Optimal l_1-controller synthesis for positive systems and its robustness properties," in *American Control Conference (ACC), 2012*. IEEE, 2012, pp. 5992–5997.

[47] A. Rantzer, "Distributed control of positive systems," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*. IEEE, 2011, pp. 6608–6611.

[48] X. Chen, J. Lam, P. Li, and Z. Shu, "l_1-induced norm and controller synthesis of positive systems," *Automatica*, vol. 49, no. 5, pp. 1377–1385, 2013.

[49] M. Ait Rami, "Solvability of static output-feedback stabilization for LTI positive systems," *Systems & Control Letters*, vol. 60, no. 9, pp. 704–708, 2011.

[50] T. Tanaka and C. Langbort, "KYP lemma for internally positive systems and a tractable class of distributed h_∞ control problems," in *American Control Conference (ACC), 2010*. IEEE, 2010, pp. 6238–6243.

[51] ——, "The bounded real lemma for internally positive systems and h_∞ structured static state feedback," *Automatic Control, IEEE Transactions on*, vol. 56, no. 9, pp. 2218–2223, 2011.

[52] J. P. Hespanha and A. S. Morse, "Switching between stabilizing controllers," *Automatica*, vol. 38, no. 11, pp. 1905–1917, 2002.

[53] J. Shamma, "The necessity of the small-gain theorem for time-varying and nonlinear systems," *Automatic Control, IEEE Transactions on*, vol. 36, no. 10, pp. 1138–1147, 1991.

[54] X. Qi, M. Khammash, and M. Salapaka, "A matlab package for multiobjective control synthesis," in *Decision and Control, 2001. Proceedings of the 40th IEEE Conference on*, vol. 4. IEEE, 2001, pp. 3991–3996.

[55] S. Jiang and P. Voulgaris, "Performance optimization of switched systems: a model matching approach," *Automatic Control, IEEE Transactions on*, vol. 54, no. 9, pp. 2058–2071, 2009.

[56] H. Chapellat and M. Dahleh, "Analysis of time-varying control strategies for optimal disturbance rejection and robustness," *Automatic Control, IEEE Transactions on*, vol. 37, no. 11, pp. 1734–1745, 1992.

[57] M. Dahleh and I. Diaz-Bobillo, *Control of uncertain systems: a linear programming approach*. Prentice-Hall, Inc., 1994.

[58] M. Khammash, "A new approach to the solution of the l_1 control problem: the scaled-q method," *Automatic Control, IEEE Transactions on*, vol. 45, no. 2, pp. 180–187, 2000.

[59] P. Voulgaris, "Optimal control of systems with delayed observation sharing patterns via input-output methods," *Systems & control letters*, vol. 50, no. 1, pp. 51–64, 2003.

[60] P. Voulgaris, M. Dahleh, and L. Valavani, "h_∞ and h_2 optimal controllers for periodic and multirate systems," *Automatica*, vol. 30, no. 2, pp. 251–263, 1994.

[61] M. Naghnaeian and P. Voulgaris, "Performance optimization over positive l_∞ cones," in *American Control Conference (ACC), 2014*. IEEE, 2014, pp. 5645–5650.

[62] N. Elia and M. A. Dahleh, *Computational methods for controller design*. Springer, 1998.

[63] M. A. Dahleh and I. J. Diaz-bobillo, "Control of uncertain systems: A linear programming approach," 1995.

[64] M. Naghnaeian and P. Voulgaris, "On model matching problems of input-output switching systems," in *51th IEEE Conference on Decision and Control Conference, 2012, Maui, Hawaii.*

[65] J. S. Shamma and K.-Y. Tu, "Set-valued observers and optimal disturbance rejection," *Automatic Control, IEEE Transactions on*, vol. 44, no. 2, pp. 253–264, 1999.

[66] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[67] T. Tanaka, "Phd dissertation, symmetric formulation of the kalman-yakubovich-popov lemma and its application to distributed control of positive systems," 2013.

[68] R. Skelton, T. Iwasaki, and D. Grigoriadis, *A unified algebraic approach to control design.* CRC Press, 1997.

[69] C. Briat, "Robust stability analysis of uncertain linear positive systems via integral linear constraints: l_1 and l_∞ gain characterizations," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on.* IEEE, 2011, pp. 6337–6342.

[70] R. B. Bapat and T. E. S. Raghavan, *Nonnegative matrices and applications.* Cambridge University Press, 1997, vol. 64.